SHEAVES OF IWASAWA MODULES, MOMENT MAPS AND THE ℓ-ADIC ELLIPTIC POLYLOGARITHM

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Abstract. In this paper we study systematically the ℓ-adic realization of the elliptic polylogarithm in the context of sheaves of Iwasawa modules. This gives an explanation of the relation between Iwasawa theory, elliptic polylogarithms and Kato's elliptic units. As an application we give a new computation of the residue of the Eisenstein symbol at the cusps and of the degeneration of the ℓ-adic Eisenstein symbol into the cyclotomic Soulé elements.

Introduction

The purpose of this paper is twofold: on the one hand it gives a new description of the étale ℓ-adic realization of the elliptic polylogarithm using systematically étale sheaves of Iwasawa modules and on the other hand it gives a new and direct proof of the main result of [HK99] about the degeneration of ℓ-adic Eisenstein classes.

The explicit description of the étale realization of the elliptic polylogarithm in terms of elliptic units was already one of the subjects in the paper [Kin01]. There we used an approach via one-motives to treat the logarithm sheaf. As the main application of the ℓ-adic elliptic polylogarithm is in the context of Iwasawa theory, it is natural to try to approach the logarithm sheaf systematically in this context. That such an approach is possible is already a suggestion in the ground-braking paper [BL94], which is one source of inspiration for this paper.

In Iwasawa theory Kato, Perrin-Riou and Colmez pointed out the usefulness to work with "Iwasawa cohomology", which is continuous Galois cohomology with values in an Iwasawa algebra. With this as a second source of inspiration, we generalize this idea to treat families of Iwasawa modules under a family of Iwasawa algebras. The main example for this is the family of Iwasawa algebras on the moduli scheme of elliptic curves, where one has in each fibre the Iwasawa algebra of the Tate module of the corresponding elliptic curve.

A third source of inspiration for this paper is the fundamental idea of Soulé [Sou81] that twisting of units can be used to produce interesting cohomology classes. It is implicit in Kato's paper [Kat93] and explicit in Colmez [Col98] that this twisting is related to the Iwasawa cohomology. We develop this further by giving a moment map at finite level for our sheaves

of Iwasawa modules and show that in the cyclotomic case one obtains the elements defined and studied by Deligne and Soulé. Work by Soulé in the CM elliptic case (and Kato's work in [Kat04]) suggests that one should carry out Soulé's twisting construction also in the modular curve case to obtain elliptic Soulé elements. These elliptic Soulé elements are none other than the ℓ-adic Eisenstein classes in [HK99].

With the general theory of sheaves of Iwasawa modules, we obtain a concrete description of the elliptic polylogarithm in terms of the norm compatible elliptic units defined and studied by Kato [Kat04]. This gives strong ties of the elliptic polylogarithm to recent developments in Iwasawa theory and also allows many explicit computations with the ℓ-adic elliptic polylogarithm.

As an application of the concrete description of the elliptic polylogarithm, we give a new proof of the residue computation on the moduli scheme for elliptic curves of the specialization of the elliptic polylogarithm along torsion sections (Theorems 7.1.2 and 7.2.1).

A second application is the evaluation of the cup-product construction used in [HK99] (and explained in this volume in [Hub]) to obtain elements in the motivic cohomology of cyclotomic fields and to prove Conjecture 6.2 in [BK90]. The approach taken here, does not need any computations of the cyclotomic polylogarithm as in [HK99]. It relies on the possibility to evaluate directly the elliptic units at the cusps.

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Contents

1. Iwasawa modules

1.1. **Iwasawa algebras and modules.** Let ℓ be a prime number and consider a profinite topological space $X = \varprojlim_r X_r$ with X_r a finite discrete set. We denote by

$$\mathcal{C}\mathrm{ont}(X, \mathbb{Z}_\ell) := \{f : X \to \mathbb{Z}_\ell \mid f \text{ continuous}\} \tag{1.1.1}$$

the space of continuous $\mathbb{Z}_\ell$-valued functions on X.

Definition 1.1.1. A $\mathbb{Z}_\ell$*-valued measure* on X is a $\mathbb{Z}_\ell$-linear map

$$\mu : \mathcal{C}\mathrm{ont}(X, \mathbb{Z}_\ell) \to \mathbb{Z}_\ell$$

and the space of all measures on X is denoted by $\Lambda(X)$. As usual we write

$$\int_X f\mu := \mu(f).$$

As every continuous function $f \in \mathcal{C}\mathrm{ont}(X, \mathbb{Z}_\ell)$ is the uniform limit of locally constant functions, we have

$$\begin{aligned}\Lambda(X) &= \varprojlim_r \mathrm{Hom}_{\mathbb{Z}_\ell}(\mathcal{C}\mathrm{ont}(X_r, \mathbb{Z}_\ell), \mathbb{Z}_\ell)\\ &= \varprojlim_r \mathbb{Z}_\ell[X_r]\\ &= \varprojlim_r \mathbb{Z}/\ell^r\mathbb{Z}[X_r],\end{aligned} \tag{1.1.2}$$

where we consider $\mathbb{Z}_\ell[X_r] := \bigoplus_{x\in X_r} \mathbb{Z}_\ell \delta_x$ as the distributions on $\mathbb{Z}_\ell$-valued functions on X_r and δ_x is the delta distribution at x. i.e., $\delta_x(f) := f(x)$ for any function $f : X_r \to \mathbb{Z}_\ell$. If $\phi : X \to Y$ is a continuous map we define

$$\phi_! : \Lambda(X) \to \Lambda(Y) \tag{1.1.3}$$

by $\phi_!\mu(f) := \mu(f \circ \phi)$. Observe that

$$\Lambda(X\times Y) = \varprojlim_r \mathbb{Z}_\ell[X_r \times Y_r] \cong \varprojlim_r (\mathbb{Z}_\ell[X_r] \otimes \mathbb{Z}_\ell[Y_r]) =: \Lambda(X)\widehat{\otimes}\Lambda(Y). \tag{1.1.4}$$

Definition 1.1.2. If $X = G = \varprojlim_r G_r$ is a profinite group, define an algebra structure on $\Lambda(G)$ by convolution of measures

$$\mu * \nu := \mathrm{mult}_!(\mu \otimes \nu), \tag{1.1.5}$$

where $\mathrm{mult} : G \times G \to G$ is the group multiplication. This algebra is the *Iwasawa algebra* of G.

Note that this ring structure makes the quotient ring $\mathbb{Z}_\ell[G_r]$ of $\Lambda(G)$ into the usual group ring. If $\gamma \in G$ we define the delta distribution δ_γ at γ to be the measure $\delta_\gamma(f) := f(\gamma)$. This induces a group homomorphism

$$\begin{aligned} \delta : G &\to \Lambda(G)^\times \\ \gamma &\mapsto \delta_\gamma. \end{aligned} \tag{1.1.6}$$

Example 1.1.3. Consider $\Lambda(\mathbb{Z}_\ell)$. By Mahler's theorem every continuous function f on $\mathbb{Z}_\ell$ can be written as

$$f(x) = \sum_{n\geq 0} a_n \binom{x}{n}$$

and hence for every measure $\mu \in \Lambda(\mathbb{Z}_\ell)$ one has

$$\mu(f) = \sum_{n\geq 0} a_n \int_{\mathbb{Z}_\ell} \binom{x}{n} \mu(x)$$

and μ is uniquely determined by the sequence of numbers $b_n := \int_{\mathbb{Z}_\ell} \binom{x}{n}\mu(x) \in \mathbb{Z}_\ell$ for $n \geq 0$. Conversely, every sequence $(b_n)_{n\geq 0}$ defines a measure μ by $\mu(f) := \sum_{n\geq 0} a_n b_n$. Thus one gets an isomorphism

$$\begin{aligned} \Lambda(\mathbb{Z}_\ell) &\xrightarrow{\cong} \mathbb{Z}_\ell[[T]]. \\ \mu &\mapsto \sum_{n\geq 0} \left(\int_{\mathbb{Z}_\ell} \binom{x}{n} \mu(x) \right) T^n \end{aligned}$$

It is useful to note that the power series associated to the measure is also given by

$$\int_{\mathbb{Z}_\ell} (1+T)^x \mu(x) \in \mathbb{Z}_\ell[[T]].$$

Assume that the space X is a torsor under G, i.e., one has a simply transitive and continuous action

$$\rho : G \times X \to X.$$

Then ρ induces a $\Lambda(G)$-module structure on $\Lambda(X)$, by

$$\Lambda(G)\widehat{\otimes}\Lambda(X) \cong \Lambda(G \times X) \xrightarrow{\rho_!} \Lambda(X)$$

and $\Lambda(X)$ is free $\Lambda(G)$-module of rank one.

We discuss three examples of profinite spaces, which are relevant for the purpose of this lecture.

Example 1.1.4. Let $N > 1$ and $t \in \mathbb{Z}/N\mathbb{Z}$ and consider for each $r \geq 0$ the exact sequence

$$0 \to \mathbb{Z}/\ell^r\mathbb{Z} \to \mathbb{Z}/\ell^r N\mathbb{Z} \xrightarrow{q_r} \mathbb{Z}/N\mathbb{Z} \to 0,$$

where q_r is reduction modulo N. We define

$$\mathbb{Z}/\ell^r\mathbb{Z}\langle t\rangle := q_r^{-1}(t) = \{x \in \mathbb{Z}/\ell^r N\mathbb{Z} \mid x \equiv t \mod N\}$$

and note that reduction modulo ℓ^{r-1} induces a map

$$\mathbb{Z}/\ell^r\mathbb{Z}\langle t\rangle \to \mathbb{Z}/\ell^{r-1}\mathbb{Z}\langle t\rangle.$$

Each $\mathbb{Z}/\ell^r\mathbb{Z}\langle t\rangle$ is an $\mathbb{Z}/\ell^r\mathbb{Z}$-torsor (which acts by translation, being a subgroup of $\mathbb{Z}/\ell^r N\mathbb{Z}$) and we define the inverse limit to be the $\mathbb{Z}_\ell$-torsor

$$\mathbb{Z}_\ell\langle t\rangle := \varprojlim_r \mathbb{Z}/\ell^r\mathbb{Z}\langle t\rangle.$$

Note that $\mathbb{Z}_\ell\langle 0\rangle = \mathbb{Z}_\ell$ and that $\Lambda(\mathbb{Z}_\ell\langle t\rangle)$ is a free rank one $\Lambda(\mathbb{Z}_\ell)$-module.

Let us define the Bernoulli measure in $\Lambda(\mathbb{Z}_\ell\langle t\rangle)$. We choose, as usual, an auxiliary $c \in \mathbb{Z}$ with $(c, \ell N) = 1$ to make the Bernoulli distribution integral (for the properties of the Bernoulli numbers we use see [Lan90, Ch. 2, §2]). Denote by $B_k(x)$ the k-th Bernoulli polynomial and consider the map

$$\begin{aligned} B^{\langle t\rangle}_{2,c,r} : \mathbb{Z}/\ell^r\mathbb{Z}\langle t\rangle &\to \mathbb{Z}/\ell^r\mathbb{Z} \\ x &\mapsto \frac{\ell^r N}{2}(c^2 B_2(\{\frac{x}{\ell^r N}\}) - B_2(\{\frac{cx}{\ell^r N}\})), \end{aligned} \tag{1.1.7}$$

where for an element $x \in \mathbb{R}/\mathbb{Z}$ we write $\{x\}$ for its representative in $[0, 1[$. Thus we have

$$B^{\langle t\rangle}_{2,c,r} \in \mathbb{Z}/\ell^r\mathbb{Z}[\mathbb{Z}/\ell^r\mathbb{Z}\langle t\rangle]$$

and by the distribution property of the Bernoulli polynomials, the $B_{2,c,r}$ are compatible in the inverse limit and give rise to a measure

$$B^{\langle t\rangle}_{2,c} := \varprojlim_r B^{\langle t\rangle}_{2,c,r} \in \Lambda(\mathbb{Z}_\ell\langle t\rangle). \tag{1.1.8}$$

Example 1.1.5. Let $\overline{K}$ be an algebraically closed field of characteristic $\neq \ell$ and consider $\mathbb{G}_m(\overline{K})$. Let $N > 1$ and $\alpha \in \mu_N(\overline{K})$ be an N-th root of unity. We let

$$\mu_{\ell^r}\langle\alpha\rangle := \{\beta \in \mu_{\ell^r N}(\overline{K}) \mid \beta^{\ell^r} = \alpha\}$$

and observe that taking the ℓ-power induces maps

$$\mu_{\ell^r}\langle\alpha\rangle \to \mu_{\ell^{r-1}}\langle\alpha\rangle.$$

Each $\mu_{\ell^r}\langle\alpha\rangle$ is an μ_{ℓ^r}-torsor. Let as usual $\mathbb{Z}_\ell(1) := \varprojlim_r \mu_{\ell^r}(\overline{K})$ then in the limit we get a $\mathbb{Z}_\ell(1)$-torsor

$$\mathbb{Z}_\ell(1)\langle\alpha\rangle := \varprojlim_r \mu_{\ell^r}\langle\alpha\rangle.$$

Again $\Lambda(\mathbb{Z}_\ell(1)\langle\alpha\rangle)$ is a free $\Lambda(\mathbb{Z}_\ell(1))$-module of rank one.

Example 1.1.6. Let $E/\overline{K}$ be an elliptic curve over an algebraically closed field of characteristic $\neq \ell$. and $t \in E(\overline{K})$. Let $[\ell^r] : E \to E$ be the ℓ^r-multiplication and define $E[\ell^r]\langle t\rangle$ by the Cartesian diagram

$$\begin{tikzcd} E[\ell^r]\langle t\rangle \arrow[r] \arrow[d] & E(\overline{K}) \arrow[d, "{[\ell^r]}"] \\ \{t\} \arrow[r] & E(\overline{K}). \end{tikzcd}$$

Note that $E[\ell^r]\langle 0\rangle = E[\ell^r]$ are the ℓ^r-torsion points and that each $E[\ell^r]\langle t\rangle$ is an $E[\ell^r]$-torsor. We let

$$T_\ell E\langle t\rangle := \varprojlim_r E[\ell^r]\langle t\rangle.$$

As before $\Lambda(T_\ell E\langle t\rangle)$ is a free $\Lambda(T_\ell E)$-module of rank one.

1.2. **The moment map.** In this section we assume that $G \cong \mathbb{Z}_\ell^d$. We write $G_r := G \otimes_{\mathbb{Z}_\ell} \mathbb{Z}/\ell^r\mathbb{Z}$ and $G_{\mathbb{Q}_\ell} := G \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$. We denote by $\mathrm{Sym}^k G$ the k-th symmetric power of the $\mathbb{Z}_\ell$-module G and write $\mathrm{Sym}^k G_r := \mathrm{Sym}^k G \otimes_{\mathbb{Z}_\ell} \mathbb{Z}/\ell^r\mathbb{Z}$ and $\mathrm{Sym}^k G_{\mathbb{Q}_\ell} := \mathrm{Sym}^k G \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$.

Let $e_1, \ldots, e_d$ be the basis of G defined by the isomorphism $G \cong \mathbb{Z}_\ell^d$ and denote by $x_i : G \to \mathbb{Z}_\ell$ the resulting i-th coordinate function. We consider $x_1, \ldots, x_d$ as elements in $G^\vee := \mathrm{Hom}_{\mathbb{Z}_\ell}(G, \mathbb{Z}_\ell)$. It is the basis dual to $e_1, \ldots, e_d$. Then $\mathrm{Sym}^k G$ has the basis

$$\{e_1^{n_1} \cdots e_d^{n_d} \mid \sum_{i=1}^d n_i = k\}. \tag{1.2.1}$$

Each monomial $x_1^{n_1} \cdots x_d^{n_d}$ with $\sum_{i=1}^d n_i = k$ can be considered as a continuous function on G and also as an element in the $\mathbb{Z}_\ell$-dual $(\mathrm{Sym}^k G)^\vee$ of $\mathrm{Sym}^k G$. In fact, if we tensor with $\mathbb{Q}_\ell$, we get that

$$\{\frac{x_1^{n_1} \cdots x_d^{n_d}}{n_1! \cdots n_d!} \mid \sum n_i = k\}. \tag{1.2.2}$$

is the basis of $\mathrm{Sym}^k G^\vee_{\mathbb{Q}_\ell}$ dual to the one of Equation (1.2.1).

Definition 1.2.1. The k-th moment map is

$$\begin{aligned} \mathrm{mom}^k : \Lambda(G) &\to \mathrm{Sym}^k G_{\mathbb{Q}_\ell} \\ \mu &\mapsto \sum_{n_1+\ldots+n_d=k} \left(\int_G x_1^{n_1}\cdots x_d^{n_d}\mu\right) \frac{e_1^{n_1}\cdots e_d^{n_d}}{n_1!\cdots n_d!}. \end{aligned} \tag{1.2.3}$$

Note that the moment map mom^k is compatible with homomorphisms $\phi : G \to H$, i.e., the diagram

$$\begin{array}{ccc} \Lambda(G) & \xrightarrow{\mathrm{mom}^k} & \mathrm{Sym}^k G_{\mathbb{Q}_\ell} \\ {\scriptstyle \phi_!}\downarrow & & \downarrow{\scriptstyle \mathrm{Sym}^k(\phi)} \\ \Lambda(H) & \xrightarrow{\mathrm{mom}^k} & \mathrm{Sym}^k H_{\mathbb{Q}_\ell} \end{array}$$

commutes.

We want to give a formula for the moment map on “finite level”. Write $\Lambda(G) = \varprojlim_r \mathbb{Z}/\ell^r\mathbb{Z}[G_r]$. Then we can write $\mu = \varprojlim_r \mu_r$ with $\mu_r \in \mathbb{Z}/\ell^r\mathbb{Z}[G_r]$. For each $\gamma \in G_r$ we can consider $\gamma^{\otimes k} \in \mathrm{Sym}^k G_r$ and we write $\mu_r(\gamma)$ for the value of μ_r on δ_γ (considered as a function on G_r).

Lemma 1.2.2. *Define a moment map at finite level by*

$$\begin{aligned} \mathrm{mom}_r^k : \mathbb{Z}/\ell^r\mathbb{Z}[G_r] &\to \mathrm{Sym}^k G_r \\ \mu_r &\mapsto \sum_{\gamma\in G_r} \mu_r(\gamma)\gamma^{\otimes k} \end{aligned}$$

then one has the formula

$$\mathrm{mom}^k(\mu) = \frac{1}{k!}(\varprojlim_r \mathrm{mom}_r^k(\mu_r)),$$

where the inverse limit is in $\mathrm{Sym}^k G \cong \varprojlim_r \mathrm{Sym}^k G_r$.

Proof. The coordinates $x_1, \ldots, x_d : G \to \mathbb{Z}_\ell$ considered as $\mathbb{Z}_\ell$-linear maps induce functions $x_i^n : G_r \to \mathbb{Z}/\ell^r\mathbb{Z}$ and by definition

$$\mu_r(x_1^{n_1}\cdots x_d^{n_d}) = \sum_{\gamma\in G_r} \mu_r(\gamma) x_1(\gamma)^{n_1}\cdots x_d(\gamma)^{n_d}.$$

If we observe that

$$k! \sum_{n_1+\ldots+n_d=k} x_1(\gamma)^{n_1}\cdots x_d(\gamma)^{n_d} \frac{e_1^{n_1}\cdots e_d^{n_d}}{n_1!\cdots n_d!} = (x_1(\gamma)e_1+\cdots+x_d(\gamma)e_d)^k = \gamma^{\otimes k}$$

we get

$$k! \sum_{n_1+\ldots+n_d=k} \mu_r(x_1^{n_1}\cdots x_d^{n_d})\frac{e_1^{n_1}\cdots e_d^{n_d}}{n_1!\cdots n_d!} = \sum_{\gamma\in G_r} \mu_r(\gamma)\gamma^{\otimes k} = \mathrm{mom}_r^k(\mu_r).$$

Taking the limit gives the desired result. ∎

We also need the moment map for $\Lambda(X)$, where X is a G-torsor of a special kind. Suppose that we have an extensions of abelian groups

$$0 \to G \to H \xrightarrow{q} T \to 0, \tag{1.2.4}$$

where T is a finite N-torsion group, and that the torsor X is of the form $X = q^{-1}(t)$ for a $t \in T$. Examples of such torsors are provided by $\mathbb{Z}_\ell\langle t\rangle$, $\mathbb{Z}_\ell(1)\langle t\rangle$ and $T_\ell E\langle t\rangle$ from 1.1.4, 1.1.5 and 1.1.6 for N-torsion points t.

Definition 1.2.3. Let $X = q^{-1}(t)$ be a G-torsor with t an N-torsion point as above and

$$\tau : X \to G$$

be the composition $X \hookrightarrow H \xrightarrow{[N]} G$ and $\tau_! : \Lambda(X) \to \Lambda(G)$ the induced morphism. The k-th moment map is

$$\mathrm{mom}_t^k : \Lambda(X) \xrightarrow{\tau_!} \Lambda(G) \xrightarrow{\frac{1}{N^k}\mathrm{mom}^k} \mathrm{Sym}^k G_{\mathbb{Q}_\ell}.$$

As for G we want a formula on "finite level" for this moment map. Let H_r be the push-out of $G \hookrightarrow H$ by $G \to G_r$ so that one has an exact sequence

$$0 \to G_r \to H_r \xrightarrow{q_r} T \to 0. \tag{1.2.5}$$

Define $X_r := q_r^{-1}(t)$ and let $\tau_{r,t}$ be the composition

$$\tau_{r,t} : X_r \hookrightarrow H_r \xrightarrow{[N]} G_r \tag{1.2.6}$$

and consider the induced map

$$\tau_{r,t!} : \mathbb{Z}/\ell^r\mathbb{Z}[X_r] \to \mathbb{Z}/\ell^r\mathbb{Z}[G_r].$$

Lemma 1.2.4. *Let X_r be a G_r-torsor of the form $X_r = q_r^{-1}(t)$ described above with $t \in T$ an N-torsion point. Let*

$$\mathrm{mom}_{r,t}^k : \mathbb{Z}/\ell^r\mathbb{Z}[X_r] \to \mathrm{Sym}^k G_r$$

be the composition $\mathrm{mom}_{r,t}^k := \mathrm{mom}_r^k \circ \tau_{r,t!}$. Then one has the formula

$$\mathrm{mom}_{r,t}^k(\mu_r) = \sum_{x \in X_r} \mu_r(x)(\tau_{r,t}(x))^{\otimes k}$$

and

$$\mathrm{mom}_t^k = \frac{1}{k!N^k} \varprojlim_r \mathrm{mom}_{r,t}^k.$$

Proof. With the notation in the proof of Lemma 1.2.2 one has

$$\tau_{r,t!}\mu_r(x_1^{n_1} \cdots x_d^{n_d}) = \mu_r((x_1 \circ \tau_{r,t})^{n_1} \cdots (x_d \circ \tau_{r,t})^{n_d}).$$

The formula

$$k! \sum_{n_1+\ldots+n_d=k} x_1(\tau_{r,t}(x)))^{n_1} \cdots x_d(\tau_{r,t}(x))^{n_d} \frac{e_1^{n_1} \cdots e_d^{n_d}}{n_1! \cdots n_d!} =$$

$$= (x_1(\tau_{r,t}(x))e_1 + \cdots + x_d(\tau_{r,t}(x))e_d)^k = \tau_{r,t}(x)^{\otimes k}$$

gives

$$k!(\tau_{r,t!}\mu_r)(x_1^{n_1}\cdots x_d^{n_d})\frac{e_1^{n_1}\cdots e_d^{n_d}}{n_1!\cdots n_d!}=\sum_{x\in X_r}\mu_r(x)(\tau_{r,t}(x))^{\otimes k}$$

and the statement of the lemma follows by taking the limit. $\square$

For later use, we compute the moments for the Bernoulli measure.

Example 1.2.5. We use the notation from Example 1.1.4. Let $t\in\mathbb{Z}/N\mathbb{Z}$ and consider $G=\mathbb{Z}_\ell$ and the torsor $X=\mathbb{Z}_\ell\langle t\rangle$. We want to compute the moments of the Bernoulli measure

$$B_{2,c}^{\langle t\rangle}:=\varprojlim_r B_{2,c,r}^{\langle t\rangle}\in\Lambda(\mathbb{Z}_\ell\langle t\rangle). \tag{1.2.7}$$

We choose $e=1\in\mathbb{Z}_\ell$ as a basis and let $x=\mathrm{id}:\mathbb{Z}_\ell\to\mathbb{Z}_\ell$ be the dual basis. By standard congruences for Bernoulli polynomials (see e.g. [Lan90, Theorem 2.1]) we have

$$\begin{aligned}\mathrm{mom}_t^k B_{2,c}^{\langle t\rangle}&=\frac{1}{k!N^k}\int_{\mathbb{Z}_\ell\langle t\rangle}x^k dB_{2,c}^{\langle t\rangle}\\&=\frac{N}{c^k(k+2)k!}(c^{k+2}B_{k+2}(\{\frac{t}{N}\})-B_{k+2}(\{\frac{ct}{N}\})).\end{aligned}\tag{1.2.8}$$

Note that if $c\equiv 1 \mod N$ we get

$$\mathrm{mom}_t^k B_{2,c}^{\langle t\rangle}=\frac{N(c^{k+2}-1)}{c^k(k+2)k!}B_{k+2}(\{\frac{t}{N}\}).\tag{1.2.9}$$

1.3. **The Iwasawa algebra $\Lambda(G)$ and the moment map.** We still assume that $G\cong\mathbb{Z}_\ell^d$ and keep the notations from the previous section.

The $\mathbb{Z}_\ell$-algebra $\Lambda(G)$ has a natural augmentation $\int_G:\Lambda(G)\to\mathbb{Z}_\ell$. We let

$$I(G):=\ker(\int_G:\Lambda(G)\to\mathbb{Z}_\ell)\tag{1.3.1}$$

be the augmentation ideal. The algebra $\Lambda(G)$ is complete with respect to $I(G)$

$$\Lambda(G)\cong\varprojlim_k\Lambda(G)/I(G)^k.$$

The augmentation ideal is generated by $\delta_{\gamma'}-\delta_\gamma$ for $\gamma,\gamma'\in G$. In particular:

Lemma 1.3.1. *The G-action on $\Lambda(G)$ given by $\delta:G\to\Lambda(G)^*$, $\gamma\mapsto\delta_\gamma$ preserves $I(G)$ and its powers $I(G)$. The induced G-action on $I(G)^k/I(G)^{k+1}$ is trivial.*

Proof. Let $\varrho\in I(G)^k$, then $(1-\delta_\gamma)\varrho\in I(G)^{k+1}$ because $1-\delta_\gamma\in I(G)$. Hence multiplication with δ_γ is trivial on $I(G)^k/I(G)^{k+1}$. $\square$

There are isomorphisms

$$\mathrm{Sym}^k G \cong I(G)^k / I(G)^{k+1} \tag{1.3.2}$$

given by $e_1^{n_1}\cdots e_d^{n_d} \mapsto (1-\delta_{e_1})^{n_1}\cdots(1-\delta_{e_d})^{n_d}$ so that the graded algebra $\bigoplus_{k\geq 0} I(G)^k/I(G)^{k+1}$ is just the symmetric algebra $\mathrm{Sym}^{\cdot} G$. Let

$$\mathfrak{U}(G_{\mathbb{Q}_\ell}) := \prod_{k\geq 0}\mathrm{Sym}^k G_{\mathbb{Q}_\ell}$$

be the completion of $\mathrm{Sym}^{\cdot} G_{\mathbb{Q}_\ell}$ with respect to its augmentation ideal $\mathfrak{I} = \prod_{k\geq 1}\mathrm{Sym}^k G_{\mathbb{Q}_\ell}$.

Proposition 1.3.2. *The map*

$$\mathrm{mom} : \Lambda(G) \to \mathfrak{U}(G_{\mathbb{Q}_\ell})$$

given by $\mu \mapsto \sum_{k\geq 0}\mathrm{mom}^k(\mu)$ *is an algebra homomorphism which preserves the augmentation and*

$$\Lambda(G) \to \Lambda(G)/I(G)^{k+1} \to \mathfrak{U}(G_{\mathbb{Q}_\ell})/\mathfrak{I}^{k+1} \to \mathrm{Sym}^k G_{\mathbb{Q}_\ell},$$

where the last map is the projection onto the k-th component, coincides with the k-th moment map mom^k*. In particular, it induces an isomorphism*

$$(\Lambda(G)/I(G)^{k+1}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \cong \mathfrak{U}(G_{\mathbb{Q}_\ell})/\mathfrak{I}^{k+1}.$$

Proof. With the chosen basis $e_1,\ldots,e_d$ of G one gets an standard ring isomorphism from Iwasawa theory

$$\begin{aligned}\Lambda(G) &\cong \mathbb{Z}_\ell[[T_1,\ldots,T_d]]\\ \delta_{e_i} &\mapsto 1+T_i\end{aligned}$$

(see Example 1.1.3). If we compose this with the ring homomorphism

$$\begin{aligned}\mathbb{Z}_\ell[[T_1,\ldots,T_d]] &\to \mathfrak{U}(G_{\mathbb{Q}_\ell})\\ T_i &\mapsto \exp(e_i)-1 = \sum_{k\geq 1}\frac{e_i^{\otimes k}}{k!}\end{aligned}$$

we get the moment map mom, which preserves the augmentation ideals. □

2. Sheaves of Iwasawa modules

2.1. **Projective systems of étale sheaves and continuous cohomology.** Let S be a noetherian scheme and assume that ℓ is invertible on S. We work with several types of projective systems of étale sheaves on S:

(1) $\mathbb{Z}_\ell$-sheaves (or ℓ-adic sheaves): Recall that a projective system of étale sheaves $\mathscr{F} = (\mathscr{F}_r)_{r\geq 0}$ on S is a $\mathbb{Z}_\ell$-sheaf, if the $\mathscr{F}_n$ are constructible $\mathbb{Z}/\ell^r\mathbb{Z}$-modules and the transition morphisms $\mathscr{F}_r \to \mathscr{F}_{r-1}$ factor through isomorphisms $\mathscr{F}_r \otimes_{\mathbb{Z}/\ell^r\mathbb{Z}} \mathbb{Z}/\ell^{r-1}\mathbb{Z} \cong \mathscr{F}_{r-1}$. We consider $\mathbb{Z}_\ell$-sheaves in the category of pro-sheaves. One has

$$\mathrm{Hom}(\mathscr{F},\mathscr{G}) = \varprojlim_r \mathrm{Hom}(\mathscr{F}_r,\mathscr{G}_r).$$

(2) $\mathbb{Q}_\ell$-sheaves: This is the quotient category of $\mathbb{Z}_\ell$-sheaves by the torsion $\mathbb{Z}_\ell$-sheaves. Thus a $\mathbb{Q}_\ell$-sheaf is given by a $\mathbb{Z}_\ell$-sheaf $\mathscr{F}$ and if one writes $\mathscr{F}\otimes\mathbb{Q}_\ell$ for $\mathscr{F}$ considered in the category of $\mathbb{Q}_\ell$-sheaves one has
$$\operatorname{Hom}(\mathscr{F}\otimes\mathbb{Q},\mathscr{G}\otimes\mathbb{Q}_\ell)=\operatorname{Hom}(\mathscr{F},\mathscr{G})\otimes_{\mathbb{Z}_\ell}\mathbb{Q}_\ell.$$
(3) Projective systems $\mathscr{F}=(\mathscr{F}_r)_{r\geq 1}$, where each $\mathscr{F}_r$ is a locally constant $\mathbb{Z}/\ell^r\mathbb{Z}$-module.
(4) Projective systems of $\mathbb{Q}_\ell$-sheaves.

If S is connected and $\overline{s}$ a geometric point of S, the category of locally constant sheaves of $\mathbb{Z}/\ell^r\mathbb{Z}$-modules is equivalent to the category of $\mathbb{Z}/\ell^r\mathbb{Z}$-modules with a continuous action of $\pi_1(S,\overline{s})$.

For projective systems $\mathscr{F}=(\mathscr{F}_r)_{r\geq 1}$ as in (1) and (3) we consider continuous étale cohomology in the sense of [Jan88]. This means
$$H^i(S,\mathscr{F})$$
is the i-th derived functor of $\mathscr{F}\mapsto\varprojlim_r H^0(S,\mathscr{F}_r)$. More generally, one defines
$$R^i\pi_*\mathscr{F}$$
for a morphism $\pi:S\to T$ to be the i-th derived functor of $\mathscr{F}\mapsto\varprojlim_r \pi_*\mathscr{F}_r$.

Of crucial importance is the following lemma:

Lemma 2.1.1. *Let $\mathscr{F}=(\mathscr{F}_r)_{r\geq 1}$ be a projective system with $H^0(S,\mathscr{F}_r)$ finite, then*
$$H^1(S,\mathscr{F})=\varprojlim_r H^1(S,\mathscr{F}_r).$$

Proof. This follows from [Jan88, Lemma 1.15, Equation (3.1)] as the $H^0(S,\mathscr{F}_r)$ satisfy the Mittag-Leffler condition. □

In the case of $\mathbb{Z}_\ell$-sheaves we consider Ext-groups
$$\operatorname{Ext}^i_S(\mathscr{F},\mathscr{G}),$$
which are the right derived functors of $\operatorname{Hom}_S(\mathscr{F},-)$. If we consider $\mathscr{F}$ and $\mathscr{G}$ in the category of $\mathbb{Q}_\ell$-sheaves, one has $\operatorname{Ext}^i_S(\mathscr{F}\otimes\mathbb{Q}_\ell,\mathscr{G}\otimes\mathbb{Q}_\ell)\cong\operatorname{Ext}^i_S(\mathscr{F},\mathscr{G})\otimes\mathbb{Q}_\ell$. We define the continuous étale cohomology of a $\mathbb{Q}_\ell$-sheaf $\mathscr{F}$ to be
$$H^i(S,\mathscr{F}\otimes\mathbb{Q}_\ell):=H^i(S,\mathscr{F})\otimes\mathbb{Q}_\ell.$$
This implies that one has a canonical map
$$H^i(S,\mathscr{F})\to H^i(S,\mathscr{F}\otimes\mathbb{Q}_\ell). \tag{2.1.1}$$

In the case (4) of projective systems of $\mathbb{Q}_\ell$-sheaves $\mathscr{F}=(\mathscr{F}_r)_{r\geq 0}$ we use the ad hoc definitions
$$\begin{aligned}H^i(S,\mathscr{F})&:=\varprojlim_r H^i(S,\mathscr{F}_r)\\ \operatorname{Ext}^i_S(\mathscr{G},\mathscr{F})&:=\varprojlim_r \operatorname{Ext}^i_S(\mathscr{G},\mathscr{F}_r),\end{aligned} \tag{2.1.2}$$

where $\mathscr{G}$ is just a $\mathbb{Q}_\ell$-sheaf.

2.2. **Étale sheaves of Iwasawa modules.** In this section we will sheafify the Iwasawa modules.

Consider a projective system of finite étale schemes $p_r : X_r \to S$ and let $X := \varprojlim_r X_r$. We denote by $\lambda_r : X_r \to X_{r-1}$ the finite étale maps in the projective system. We denote by $\mathscr{X}_r$ the étale sheaf associated to X_r and define an étale sheaf on S by

$$\mathbb{Z}/\ell^r\mathbb{Z}[\mathscr{X}_r] := p_{r*}\mathbb{Z}/\ell^r\mathbb{Z}. \tag{2.2.1}$$

The trace with respect to λ_r induces a map $\lambda_{r*}\mathbb{Z}/\ell^r\mathbb{Z} \to \mathbb{Z}/\ell^r\mathbb{Z}$, which gives rise to

$$p_{r*}\mathbb{Z}/\ell^r\mathbb{Z} = p_{r-1*}\lambda_{r*}\mathbb{Z}/\ell^r\mathbb{Z} \to p_{r-1*}\mathbb{Z}/\ell^r\mathbb{Z} \to p_{r-1*}\mathbb{Z}/\ell^{r-1}\mathbb{Z},$$

where the last map is reduction modulo ℓ^{r-1}.

Definition 2.2.1. Define a projective system of étale sheaves on S by

$$\Lambda(\mathscr{X}) := (\mathbb{Z}/\ell^r\mathbb{Z}[\mathscr{X}_r])_{r\geq 1},$$

with the above transition maps.

Note that $\Lambda(\mathscr{X})$ is not an $\mathbb{Z}_\ell$-sheaf in general.

This construction is functorial in the following sense. Let $(f_r : X_r \to Y_r)_{r\geq 0}$ be a morphism of projective systems with each f_r separated étale. Then the trace map induces

$$f_{r!} : \mathbb{Z}/\ell^r\mathbb{Z}[\mathscr{X}_r] \to \mathbb{Z}/\ell^r\mathbb{Z}[\mathscr{Y}_r] \tag{2.2.2}$$

and hence a map $f_! : \Lambda(\mathscr{X}) \to \Lambda(\mathscr{Y})$.

To explain the relation with the construction in section 1.1 let us choose a geometric point $\overline{s} : \mathrm{Spec}\overline{K} \to S$ and let $\mathscr{X}_{r,\overline{s}}$ and $\mathscr{X}_{\overline{s}}$ be the stalk of $\mathscr{X}_r$ respectively $\mathscr{X}$ over $\overline{s}$. We consider $\mathscr{X}_{r,\overline{s}}$ as a finite set with a continuous Galois action. Immediately from the definitions we have:

Lemma 2.2.2. *The stalk of $\Lambda(\mathscr{X})$ in $\overline{s}$ is*

$$\Lambda(\mathscr{X})_{\overline{s}} \cong \Lambda(\mathscr{X}_{\overline{s}}) = \varprojlim_r \mathbb{Z}/\ell^r\mathbb{Z}[\mathscr{X}_{r,\overline{s}}].$$

If each $X_r = G_r$ is a finite étale group scheme over S, so that $G := \varprojlim_r G_r$ is a pro-étale group scheme, the sheaves $\mathbb{Z}/\ell^r\mathbb{Z}[\mathscr{G}_r]$ become sheaves of algebras and the results from section 1.1 carry over mutatis mutandis.

Lemma 2.2.3. *One has an isomorphism*

$$\mathbb{Z}/\ell^r\mathbb{Z}[\mathscr{G}_r \times_S \mathscr{G}_r] \cong \mathbb{Z}/\ell^r\mathbb{Z}[\mathscr{G}_r] \otimes \mathbb{Z}/\ell^r\mathbb{Z}[\mathscr{G}_r]$$

and the group multiplication $\mathrm{mult} : G_r \times_S G_r \to G_r$ *induces the ring structure*

$$\mathrm{mult}_! : \mathbb{Z}/\ell^r\mathbb{Z}[\mathscr{G}_r] \otimes \mathbb{Z}/\ell^r\mathbb{Z}[\mathscr{G}_r] \to \mathbb{Z}/\ell^r\mathbb{Z}[\mathscr{G}_r].$$

Proof. Let $p : G_r \to S$ be the structure map. The first statement follows from

$$(p \times p)_* \mathbb{Z}/\ell^r\mathbb{Z} \cong p_*\mathbb{Z}/\ell^r\mathbb{Z} \otimes p_*\mathbb{Z}/\ell^r\mathbb{Z}$$

and the second from

$$(p \times p)_* \mathbb{Z}/\ell^r\mathbb{Z} = p_* \circ \mathrm{mult}_* \mathbb{Z}/\ell^r\mathbb{Z} \xrightarrow{\mathrm{tr}} p_*\mathbb{Z}/\ell^r\mathbb{Z}.$$

□

2.3. **Examples of sheaves of Iwasawa modules.** We discuss two examples, which will be important later.

Example 2.3.1. Consider a family of elliptic curves $\pi : \mathcal{E} \to S$ with unit section $e : S \to \mathcal{E}$ and the ℓ^r-multiplication map $[\ell^r] : \mathcal{E} \to \mathcal{E}$. For each section $t \in \mathcal{E}(S)$ let $\mathcal{E}[\ell^r]\langle t\rangle$ be the fibre product

$$\begin{array}{ccc} \mathcal{E}[\ell^r]\langle t\rangle & \longrightarrow & \mathcal{E} \\ {\scriptstyle p_r\langle t\rangle}\downarrow & & \downarrow{\scriptstyle [\ell^r]} \\ S & \xrightarrow{t} & \mathcal{E}. \end{array} \tag{2.3.1}$$

This is a finite étale scheme over S and the ℓ-multiplication induces maps $[\ell] : \mathcal{E}[\ell^r]\langle t\rangle \to \mathcal{E}[\ell^{r-1}]\langle t\rangle$. Note that $\mathcal{E}[\ell^r]\langle e\rangle = \mathcal{E}[\ell^r]$ and that each $\mathcal{E}[\ell^r]\langle t\rangle$ is a torsor under $\mathcal{E}[\ell^r]$. We will consider $\mathcal{E}[\ell^r]\langle t\rangle$ also as an étale sheaf over S and then denote it by $\mathscr{H}_{\mathbb{Z}/\ell^r\mathbb{Z}}\langle t\rangle$ in the case $t \neq e$ and by $\mathscr{H}_{\mathbb{Z}/\ell^r\mathbb{Z}}$ if $t = e$. We define

$$\mathscr{H}_{\mathbb{Z}_\ell}\langle t\rangle := (\mathscr{H}_{\mathbb{Z}/\ell^r\mathbb{Z}}\langle t\rangle)_{r\geq 1} \tag{2.3.2}$$

with respect to the transition maps given by ℓ-multiplication. Similarly, we write

$$\mathscr{H}_{\mathbb{Z}_\ell} := (\mathscr{H}_{\mathbb{Z}/\ell^r\mathbb{Z}})_{r\geq 1}. \tag{2.3.3}$$

The stalk of $\mathscr{H}_{\mathbb{Z}_\ell}$ at $\overline{s}$ is nothing but the usual Tate module $T_\ell\mathcal{E}_{\overline{s}}$ of the elliptic curve $\mathcal{E}_{\overline{s}}$ (the fibre of $\mathcal{E}$ over $\overline{s}$). One also has $\mathscr{H}_{\mathbb{Z}/\ell^r\mathbb{Z}}\langle t\rangle_{\overline{s}} = \mathcal{E}_{\overline{s}}[\ell^r]\langle t_{\overline{s}}\rangle$, where $t_{\overline{s}}$ is the section of $\mathcal{E}_{\overline{s}}$ induced by t. In particular,

$$\Lambda(\mathscr{H}_{\mathbb{Z}_\ell})_{\overline{s}} \cong \Lambda(T_\ell\mathcal{E}_{\overline{s}}) \qquad \text{and} \qquad \Lambda(\mathscr{H}_{\mathbb{Z}_\ell}\langle t\rangle)_{\overline{s}} \cong \Lambda(T_\ell\mathcal{E}_{\overline{s}}\langle t_{\overline{s}}\rangle). \tag{2.3.4}$$

Note that $\Lambda(\mathscr{H}_{\mathbb{Z}_\ell}\langle t\rangle)$ is a free $\Lambda(\mathscr{H}_{\mathbb{Z}_\ell})$-module of rank one. If we compare this with example 1.1.6, we see that we have sheafified the construction given there.

In the above example we have varied the elliptic curve, now we also want to vary the section t.

Example 2.3.2. Consider $\mathcal{E}_r = \mathcal{E}$ as a finite étale $\mathcal{E}$-scheme via the ℓ^r-multiplication map $[\ell^r] : \mathcal{E}_r \to \mathcal{E}$. Define an étale sheaf on $\mathcal{E}$ by

$$\mathcal{L}_{\mathbb{Z}/\ell^r\mathbb{Z}} := [\ell^r]_*\mathbb{Z}/\ell^r\mathbb{Z}. \tag{2.3.5}$$

The trace with respect to $[\ell] : \mathcal{E}_r \to \mathcal{E}_{r-1}$ composed with reduction modulo ℓ^{r-1} induces $\mathcal{L}_{\mathbb{Z}/\ell^r\mathbb{Z}} \to \mathcal{L}_{\mathbb{Z}/\ell^{r-1}\mathbb{Z}}$. We define

$$\mathcal{L}_{\mathbb{Z}_\ell} := (\mathcal{L}_{\mathbb{Z}/\ell^r\mathbb{Z}})_{r\geq 1}. \tag{2.3.6}$$

Let $t : S \to \mathcal{E}$ be a section and recall that $p_r\langle t\rangle : \mathcal{E}[\ell^r]\langle t\rangle \to S$ is the pull-back of $\mathcal{E}_r$ by t. By proper base change $t^*[\ell^r]_*\mathbb{Z}/\ell^r\mathbb{Z} \cong p_r\langle t\rangle_*\mathbb{Z}/\ell^r\mathbb{Z}$ so that

$$t^*\mathcal{L}_{\mathbb{Z}_\ell} \cong \Lambda(\mathscr{H}_{\mathbb{Z}_\ell}\langle t\rangle). \tag{2.3.7}$$

We can also compute the stalks of $\mathcal{L}_{\mathbb{Z}_\ell}$. Let $\overline{s}$ be a geometric point of S and $\overline{t} := t(\overline{s})$ the image of $\overline{s}$ in $\mathcal{E}$. Then the stalk of $\mathcal{L}_{\mathbb{Z}_\ell}$ at $\overline{t}$ is

$$\mathcal{L}_{\mathbb{Z}_\ell,\overline{t}} \cong (t^*\mathcal{L}_{\mathbb{Z}_\ell})_{\overline{s}} \cong \Lambda(\mathscr{H}_{\mathbb{Z}_\ell}\langle t\rangle_{\overline{s}}) \cong \Lambda(T_\ell\mathcal{E}_{\overline{s}}\langle t_{\overline{s}}\rangle), \tag{2.3.8}$$

where $\mathcal{E}_{\overline{s}}$ is the base change of $\mathcal{E}$ to $\overline{s}$ as before.

Definition 2.3.3. The sheaf $\mathcal{L}_{\mathbb{Z}_\ell}$ is called the *integral ℓ-adic logarithm sheaf.*

We will see in Theorem 8.2.4 how $\mathcal{L}_{\mathbb{Z}_\ell}$ is related to the logarithm sheaf $\mathscr{L}\mathrm{og}_{\mathbb{Z}_\ell}$ and in which sense this definition is justified.

3. The moment map for étale sheaves

3.1. **The sheafified moment map.** In this section we describe a sheaf version of the moment maps from Definitions 1.2.1 and 1.2.3.

Let $G_r \to S$ be a finite étale group scheme. We *assume* that étale locally

$$G_r \cong (\mathbb{Z}/\ell^r\mathbb{Z})^d$$

for $d \geq 1$ and let $\mathscr{G}_r$ be the associated étale sheaf on S. We consider G_r-torsors X_r of the following kind: Let

$$0 \to G_r \to H_r \xrightarrow{q} T \to 0$$

be an exact sequence of finite étale commutative group schemes, where T is an N-torsion group. We consider G_r-torsors $p : X_r \to S$ given by a cartesian square

$$\begin{tikzcd} X_r : \arrow[r, equal] & G_r\langle t\rangle \arrow[r] \arrow[d, "p"'] & H_r \arrow[d, "q"] \\ & S \arrow[r, "t"] & T \end{tikzcd} \tag{3.1.1}$$

Definition 3.1.1. Let $\tau_{r,t} : X_r \to G_r$ be the composition

$$\tau_{r,t} : X_r \hookrightarrow H_r \xrightarrow{[N]} G_r. \tag{3.1.2}$$

This defines a section $\tau_{r,t} \in H^0(X_r, p^*\mathscr{G}_r)$ and a sheaf homomorphism $\tau_{r,t!} : \mathbb{Z}/\ell^r\mathbb{Z}[\mathscr{X}_r] \to \mathbb{Z}/\ell^r\mathbb{Z}[\mathscr{G}_r]$. Let

$$\tau_{r,t}^{\otimes k} \in H^0(X_r, p^*\mathrm{Sym}^k\mathscr{G}_r) \tag{3.1.3}$$

be the k-th symmetric power of $\tau_{r,t}$, where $\mathrm{Sym}^k\mathscr{G}_r := \mathrm{Sym}^k_{\mathbb{Z}/\ell^r\mathbb{Z}}\mathscr{G}_r$.

We will view $\tau_{r,t}^{\otimes k}$ also as a map of sheaves

$$\tau_{r,t}^{\otimes k}:\mathbb{Z}/\ell^r\mathbb{Z}\to p_*p^*\mathrm{Sym}^k\mathscr{G}_r. \tag{3.1.4}$$

Recall that for sheaves $\mathscr{F},\mathscr{H}$ on G_r one has the morphism (given by the projection formula and adjunction)

$$p_!\mathscr{F}\otimes p_*\mathscr{H}\cong p_!(\mathscr{F}\otimes p^*p_*\mathscr{H})\to p_!(\mathscr{F}\otimes\mathscr{H}) \tag{3.1.5}$$

and that $p_!=p_*$ as p is finite.

Definition 3.1.2. Let $X_r=G_r\langle t\rangle$ be as above. The sheafified moment map

$$\mathrm{mom}_{r,t}^k:\mathbb{Z}/\ell^r\mathbb{Z}[\mathscr{X}_r]\to\mathrm{Sym}^k\mathscr{G}_r$$

is the composition

$$\begin{aligned}p_*\mathbb{Z}/\ell^r\mathbb{Z}\xrightarrow{\mathrm{id}\otimes\tau_{r,t}^{\otimes k}} p_*\mathbb{Z}/\ell^r\mathbb{Z}\otimes p_*p^*\mathrm{Sym}^k\mathscr{G}_r\xrightarrow{(3.1.5)}\\ \to p_*(\mathbb{Z}/\ell^r\mathbb{Z}\otimes p^*\mathrm{Sym}^k\mathscr{G}_r)\cong p_*p^*\mathrm{Sym}^k\mathscr{G}_r\xrightarrow{\mathrm{tr}}\mathrm{Sym}^k\mathscr{G}_r,\end{aligned}$$

where tr is the trace map.

Let $G:=\varprojlim_r G_r$ be a pro-étale group scheme, with G_r as above and such that $G_r\otimes\mathbb{Z}/\ell^{r-1}\mathbb{Z}\cong G_{r-1}$. Suppose that $X:=\varprojlim_r X_r$ is a corresponding system of G_r-torsors such that the push-out of X_r via $G_r\to G_{r-1}$ is X_{r-1}. Then the moment maps $\mathrm{mom}_{r,t}^k:\mathbb{Z}/\ell^r\mathbb{Z}[\mathscr{X}_r]\to\mathrm{Sym}^k\mathscr{G}_r$ are compatible in the projective system.

Definition 3.1.3. We denote by

$$\widetilde{\mathrm{mom}}_t^k=\varprojlim_r\mathrm{mom}_{r,t}^k:\Lambda(\mathscr{X})\to\mathrm{Sym}^k\mathscr{G}$$

the resulting map on the projective systems.

The moment map is functorial in the following sense.

Lemma 3.1.4. *Given a separated étale homomorphism of group schemes*

$$\varphi:G_r\to G'_r$$

and a map $\psi:X_r\to X'_r$ of torsors compatible with φ there is a commutative diagram

$$\begin{tikzcd}\mathbb{Z}/\ell^r\mathbb{Z}[\mathscr{X}_r] \arrow[r, "\psi_!"] \arrow[d, "\mathrm{mom}_{r,t}^k"'] & \mathbb{Z}/\ell^r\mathbb{Z}[\mathscr{X}'_r] \arrow[d, "\mathrm{mom}_{r,t}^k"] \\ \mathrm{Sym}^k\mathscr{G}_r \arrow[r, "\mathrm{Sym}^k\varphi"] & \mathrm{Sym}^k\mathscr{G}'_r.\end{tikzcd}$$

Proof. This follows directly from the Definition 3.1.2 of $\mathrm{mom}_{r,t}^k$. □

Let us show that on stalks the moment map coincides with the one defined in Lemma 1.2.4.

Lemma 3.1.5. *Let $\overline{s}$ be a geometric point of S, then*

$$(\mathrm{mom}^k_{r,t})_{\overline{s}} : \mathbb{Z}/\ell^r\mathbb{Z}[\mathscr{X}_{r,\overline{s}}] \to \mathrm{Sym}^k\mathscr{G}_{r,\overline{s}}$$

coincides with the moment map $\mathrm{mom}^k_{r,t}$ *defined in Lemma 1.2.4.*

Proof. We have $(p_*\mathbb{Z}/\ell^r\mathbb{Z})_{\overline{s}} = \mathbb{Z}/\ell^r\mathbb{Z}[\mathscr{X}_{r,\overline{s}}]$ and let

$$\mu_r = \sum_{\overline{x}\in\mathscr{G}_{r,\overline{s}}} m_{\overline{x}}\delta_{\overline{x}} \in \mathbb{Z}/\ell^r\mathbb{Z}[\mathscr{X}_{r,\overline{s}}]$$

be an element. Let $\mathrm{Sym}^k\mathscr{G}_{r,\overline{s}}[\mathscr{X}_{r,\overline{s}}] := p_*p^*\mathrm{Sym}^k\mathscr{G}_{r,\overline{s}}$, then

$$(p_*\mathbb{Z}/\ell^r\mathbb{Z} \otimes p_*p^*\mathrm{Sym}^k\mathscr{G}_r)_{\overline{s}} = \mathbb{Z}/\ell^r\mathbb{Z}[\mathscr{X}_{r,\overline{s}}] \otimes \mathrm{Sym}^k\mathscr{G}_{r,\overline{s}}[\mathscr{X}_{r,\overline{s}}]$$

and the image of μ_r under $\mathrm{id}\otimes\tau^{\otimes k}_{r,t}$ is given by

$$(\sum_{\overline{x}\in\mathscr{X}_{r,\overline{s}}} m_{\overline{x}}\delta_{\overline{x}}) \otimes (\sum_{\overline{y}\in\mathscr{X}_{r,\overline{s}}} \tau^{\otimes k}_{r,t}(\overline{y})\delta_{\overline{y}}). \tag{3.1.6}$$

The homomorphism

$$\mathbb{Z}/\ell^r\mathbb{Z}[\mathscr{X}_{r,\overline{s}}] \otimes \mathrm{Sym}^k\mathscr{G}_{r,\overline{s}}[\mathscr{X}_{r,\overline{s}}] \xrightarrow{(3.1.5)} \mathrm{Sym}^k\mathscr{G}_{r,\overline{s}}[\mathscr{X}_{r,\overline{s}}]$$

maps the element in (3.1.6) to

$$\sum_{\overline{x}\in\mathscr{X}_{r,\overline{s}}} m_{\overline{x}}\tau^{\otimes k}_{r,t}(\overline{x})\delta_{\overline{x}} = \sum_{\overline{x}\in\mathscr{X}_{r,\overline{s}}} \mu_r(\overline{x})\tau^{\otimes k}_{r,t}(\overline{x})\delta_{\overline{x}}$$

and the trace of this is

$$\sum_{\overline{x}\in\mathscr{X}_{r,\overline{s}}} \mu_r(\overline{x})\tau^{\otimes k}_{r,t}(\overline{x}) = \sum_{\overline{x}\in\mathscr{X}_{r,\overline{s}}} \mu_r(\overline{x})([N]\overline{x})^{\otimes k}$$

which is the formula in Lemma 1.2.4. □

The next proposition is crucial for the comparison between the twisting construction of Soulé and the moment map.

Proposition 3.1.6. *The homomorphism*

$$\mathrm{mom}^k_{r,t} : H^1(S, \mathbb{Z}/\ell^r\mathbb{Z}[\mathscr{X}_r](1)) \to H^1(S, \mathrm{Sym}^k\mathscr{G}_r(1))$$

induced by the moment map $\mathrm{mom}^k_{r,t}$ *coincides with*

$$H^1(X_r, \mathbb{Z}/\ell^r\mathbb{Z}(1)) \xrightarrow{\cup\tau_r^{\otimes k}} H^1(X_r, p^*\mathrm{Sym}^k\mathscr{G}_r(1)) \xrightarrow{\mathrm{tr}} H^1(S, \mathrm{Sym}^k\mathscr{G}_r(1)),$$

where the first map is the cup-product with $\tau^{\otimes k}_{r,t} \in H^0(X_r, p^*\mathrm{Sym}^k\mathscr{G}_r)$ *and the second map is the trace map with respect to* $p : X_r \to S$.

Proof. This follows from the commutative diagram

$$\begin{tikzcd}
H^1(X_r, \mathbb{Z}/\ell^r\mathbb{Z}(1)) \times H^0(X_r, p^*\mathrm{Sym}^k\mathscr{G}_r) \arrow[r, "\cup"] \arrow[dd, "\cong"] & H^1(X_r, p^*\mathrm{Sym}^k\mathscr{G}_r(1)) \arrow[d] \\
& H^1(S, p_*(\mathbb{Z}/\ell^r\mathbb{Z} \otimes p^*\mathrm{Sym}^k\mathscr{G}_r(1))) \\
H^1(S, p_*\mathbb{Z}/\ell^r\mathbb{Z}(1)) \times H^0(S, p_*p^*\mathrm{Sym}^k\mathscr{G}_r) \arrow[r, "\cup"] & H^1(S, p_*\mathbb{Z}/\ell^r\mathbb{Z} \otimes p_*p^*\mathrm{Sym}^k\mathscr{G}_r(1)), \arrow[u, "(3.1.5)"']
\end{tikzcd}$$

where the unlabelled vertical arrows are the edge morphisms of the Leray spectral sequence for Rp_*. □

3.2. **The Kummer map and sheaves of Iwasawa modules.** Let us recall the *Kummer map* for a scheme T on which ℓ is invertible. Consider the exact sequence of étale sheaves on T

$$1 \to \mu_{\ell^r} \to \mathbb{G}_m \xrightarrow{[\ell^r]} \mathbb{G}_m \to 1.$$

Definition 3.2.1. The *Kummer map* into the first étale cohomology is the boundary map for the above exact sequence

$$\partial_r : \mathbb{G}_m(T) \to H^1(T, \mathbb{Z}/\ell^r\mathbb{Z}(1)). \tag{3.2.1}$$

Note that an element in $\mathbb{G}_m(T)$ is just an invertible function on T. Let $G\langle t\rangle = \varprojlim_r G_r\langle t\rangle$ be an projective system of torsors over S as in Diagram (3.1.1). For each $G_r\langle t\rangle$ we have an isomorphism (induced from the edge morphism of the Leray spectral sequence for $Rp_r\langle t\rangle_*$)

$$H^1(G_r\langle t\rangle, \mathbb{Z}/\ell^r\mathbb{Z}(1)) \cong H^1(S, \mathbb{Z}/\ell^r\mathbb{Z}[\mathscr{G}_r\langle t\rangle](1))$$

because $\mathbb{Z}/\ell^r\mathbb{Z}[\mathscr{G}_r\langle t\rangle] = p_r\langle t\rangle_*\mathbb{Z}/\ell^r\mathbb{Z}$, where $p_r\langle t\rangle_* : G_r\langle t\rangle \to S$ is the structure map and $p_r\langle t\rangle_*$ is finite. The transition morphisms $\lambda_r : G_r\langle t\rangle \to G_{r-1}\langle t\rangle$ induce $\lambda_{r,!} : \mathbb{Z}/\ell^r\mathbb{Z}[\mathscr{G}_r\langle t\rangle] \to \mathbb{Z}/\ell^{r-1}\mathbb{Z}[\mathscr{G}_{r-1}\langle t\rangle]$. As $\lambda_{r,!}$ is induced by the trace map, we get the following compatibility:

Lemma 3.2.2. *The diagram*

$$\begin{tikzcd}
H^1(S, \mathbb{Z}/\ell^r\mathbb{Z}[\mathscr{G}_r\langle t\rangle](1)) \arrow[r, "\cong"] \arrow[d, "\lambda_{r,!}"'] & H^1(G_r\langle t\rangle, \mathbb{Z}/\ell^r\mathbb{Z}(1)) \arrow[d, "\lambda_{r,*}"] \\
H^1(S, \mathbb{Z}/\ell^{r-1}\mathbb{Z}[\mathscr{G}_{r-1}]\langle t\rangle](1)) \arrow[r, "\cong"] & H^1(G_{r-1}\langle t\rangle, \mathbb{Z}/\ell^{r-1}\mathbb{Z}(1)),
\end{tikzcd}$$

where $\lambda_{r,*}$ is the trace map, commutes. Taking the projective limit, one gets using Lemma 2.1.1

$$H^1(S, \Lambda(\mathscr{G}\langle t\rangle)(1)) \cong \varprojlim_r H^1(G_r\langle t\rangle, \mathbb{Z}/\ell^r\mathbb{Z}(1)).$$

Suppose that we have a norm-compatible system of invertible functions on the $G_r\langle t\rangle$. This means we have for each $r \geq 1$ a function $f_r \in \mathbb{G}_m(G_r\langle t\rangle)$ such that

$$\lambda_{r,*}(f_r) = f_{r-1},$$

where $\lambda_{r,*}$ is the norm map with respect to λ_r. Then one has a commutative diagram

$$\begin{array}{ccc} \mathbb{G}_m(G_r\langle t\rangle) & \xrightarrow{\partial_r} & H^1(G_r\langle t\rangle, \mathbb{Z}/\ell^r\mathbb{Z}(1)) \\ \downarrow{\scriptstyle\lambda_{r,*}} & & \downarrow{\scriptstyle\lambda_{r,*}} \\ \mathbb{G}_m(G_{r-1}\langle t\rangle) & \xrightarrow{\partial_{r-1}} & H^1(G_{r-1}\langle t\rangle, \mathbb{Z}/\ell^{r-1}\mathbb{Z}(1)) \end{array} \tag{3.2.2}$$

and one can define

$$\varprojlim_r (\partial_r(f_r)) \in H^1(S, \Lambda(\mathscr{G}\langle t\rangle)(1)). \tag{3.2.3}$$

4. Cyclotomic Soulé-Deligne elements

4.1. Modified cyclotomic Soulé-Deligne elements. We review the cyclotomic elements defined by Soulé [Sou81] and Deligne [Del89] from our perspective. Let $N > 1$ and $c \in \mathbb{Z}$ be an integer prime to ℓN and let $S = \operatorname{Spec}\mathbb{Q}(\mu_N)$ be the spectrum of the field of N-th roots of unity.

To define the modified Soulé-Deligne elements consider the schemes $\mu_{\ell^r N} \subset \mathbb{G}_m$ over S. For each $\alpha \in \mu_N(S)$ we define $\mu_{\ell^r}\langle\alpha\rangle$ by the cartesian diagram

$$\begin{array}{ccccc} \mu_{\ell^r}\langle\alpha\rangle & \longrightarrow & \mu_{\ell^r N}\setminus\{1\} & \longrightarrow & \mathbb{G}_m\setminus\{1\} \\ \downarrow{\scriptstyle p_r\langle\alpha\rangle} & & \downarrow{\scriptstyle[\ell^r]} & & \downarrow{\scriptstyle[\ell^r]} \\ S & \xrightarrow{\alpha} & \mu_N & \longrightarrow & \mathbb{G}_m. \end{array}$$

Note that $\mu_{\ell^r}\langle 1\rangle = \mu_{\ell^r}\setminus\{1\}$ contrary to what was defined earlier. On $\mathbb{G}_m\setminus\{1\}$ we have the function $1-z : \mathbb{G}_m\setminus\{1\} \to \mathbb{G}_m$, which by restriction to $\mu_{\ell^r}\langle\alpha\rangle$ gives an invertible function

$$1 - z \in \mathbb{G}_m(\mu_{\ell^r}\langle\alpha\rangle)$$

and hence by the Kummer map an element

$$\partial_r(1-z) \in H^1(\mu_{\ell^r}\langle\alpha\rangle, \mathbb{Z}/\ell^r\mathbb{Z}(1)). \tag{4.1.1}$$

The function $1-z$ is *norm-compatible*, which means that

$$[\ell^r]_*(1-z) = 1-z, \tag{4.1.2}$$

where $[\ell^r]_*$ is the norm map with respect to the ℓ^r-power map on $\mathbb{G}_m$.

As in Definition 3.1.1 we have a section $\tau_{r,\alpha} \in H^0(\mu_{\ell^r}\langle\alpha\rangle, \mathbb{Z}/\ell^r\mathbb{Z}(1))$ defined by

$$\tau_{r,\alpha} : \mu_{\ell^r}\langle\alpha\rangle \hookrightarrow \mu_{\ell^r N} \xrightarrow{[N]} \mu_{\ell^r}.$$

Taking the k-th tensor power of this section gives

$$\tau_{r,\alpha}^{\otimes k} \in H^0(\mu_{\ell^r}\langle\alpha\rangle, \mathbb{Z}/\ell^r\mathbb{Z}(k)). \tag{4.1.3}$$

Definition 4.1.1. Let $p_r\langle\alpha\rangle : \mu_{\ell^r}\langle\alpha\rangle \to S$ be the structure map and define

$$\widetilde{c}_{k+1,r}(\alpha) := p_r\langle\alpha\rangle_*(\partial_r(1-z) \cup \tau_{r,\alpha}^{\otimes k}) \in H^1(S, \mathbb{Z}/\ell^r\mathbb{Z}(k+1)). \tag{4.1.4}$$

Note that over $T := \mathrm{Spec}\mathbb{Q}(\mu_{\ell^r N})$ the section $\tau_{r,\alpha}^{\otimes k}$ is given by $\beta \mapsto (\beta^N)^{\otimes k}$ for $\beta \in \mu_{\ell^r}\langle\alpha\rangle(T)$. Moreover, one has

$$\mu_{\ell^r}\langle\alpha\rangle(T) = \{\beta \in \mu_{\ell^r N}(T) \setminus \{1\} \mid \beta^{\ell^r} = \alpha\}.$$

It follows that the pull-back of $\widetilde{c}_{k+1,r}(\alpha)$ to $H^1(T, \mathbb{Z}/\ell^r\mathbb{Z}(k+1))$ is given explicitly by

$$\widetilde{c}_{k+1,r}(\alpha) = \sum_{\beta \in \mu_{\ell^r}\langle\alpha\rangle(T)} \partial_r(1-\beta) \cup (\beta^N)^{\otimes k}. \tag{4.1.5}$$

It is an important insight of Soulé that these elements are compatible for different $r \geq 1$ so that one can pass to the inverse limit:

Lemma 4.1.2 (Soulé)**.** *Denote by*

$$\mathrm{red}_r : H^1(S, \mathbb{Z}/\ell^r\mathbb{Z}(k+1)) \to H^1(S, \mathbb{Z}/\ell^{r-1}\mathbb{Z}(k+1))$$

the reduction map modulo ℓ^{r-1}. Then $\mathrm{red}_r(c_{k+1,r}(\alpha)) = c_{k+1,r-1}(\alpha)$.

Proof. Let $[\ell] : \mu_{\ell^r}\langle\alpha\rangle \to \mu_{\ell^{r-1}}\langle\alpha\rangle$ be the map induced from the ℓ-power map. Then the reduction modulo ℓ^{r-1} of $\tau_{r,\alpha}^{\otimes k}$ is $\mathrm{red}_r(\tau_{r,\alpha}^{\otimes k}) = [\ell]^*(\tau_{r-1,\alpha}^{\otimes k})$. By the norm-compatibility of the function $1-z$ we have $[\ell]_*\mathrm{red}_r(\partial_r(1-z)) = \partial_{r-1}(1-z)$, so that

$$\begin{aligned}
\mathrm{red}_r(c_{k+1,r}(\alpha)) &= \mathrm{red}_r p_r\langle\alpha\rangle_*(\partial_r(1-z) \cup \tau_{r,\alpha}^{\otimes k}) \\
&= p_{r-1}\langle\alpha\rangle_*[\ell]_*(\mathrm{red}_r(\partial_r(1-z)) \cup \mathrm{red}_r\tau_{r,\alpha}^{\otimes k}) \\
&= p_{r-1}\langle\alpha\rangle_*[\ell]_*(\mathrm{red}_r(\partial_r(1-z)) \cup [\ell]^*(\tau_{r-1,\alpha}^{\otimes k})) \\
&= p_{r-1}\langle\alpha\rangle_*(\partial_{r-1}(1-z) \cup \tau_{r-1,\alpha}^{\otimes k}) \\
&= c_{k+1,r-1}(\alpha).
\end{aligned}$$

by the projection formula. □

Definition 4.1.3. For $\alpha \in \mu_N(S)$ and $k \in \mathbb{Z}$ the *modified cyclotomic Soulé-Deligne element* is

$$\widetilde{c}_{k+1}(\alpha) := \varprojlim_r \widetilde{c}_{k+1,r}(\alpha) \in H^1(S, \mathbb{Z}_\ell(k+1)).$$

Moreover, for a function $\psi : \mu_N(S) \to \mathbb{Z}_\ell$ we let

$$\widetilde{c}_{k+1}(\psi) := \sum_{\alpha \in \mu_N} \psi(\alpha)\widetilde{c}_{k+1}(\alpha).$$

4.2. **Soulé-Deligne elements as moments.** In this section we will show how the Soulé-Deligne elements can be obtained as moments from a cohomology class

$$\mathcal{S}_c^{\langle\alpha\rangle} \in H^1_{\mathrm{cont}}(S, \Lambda(\mathbb{Z}_\ell(1)\langle\alpha\rangle)(1)).$$

To define $\mathcal{S}_c^{\langle\alpha\rangle}$, we start with a slightly different function, which will be closer to what we get out of the elliptic polylogarithm. Consider for $c \in \mathbb{Z}$ with $(c, \ell N) = 1$ the function

$$_c\Xi : \mathbb{G}_m \setminus \mu_c \to \mathbb{G}_m \tag{4.2.1}$$

defined by $z \mapsto \frac{(1-z)^{c^2}}{1-z^c}$. From the norm-compatibility of $1-z$ it follows that

$$[\ell^r]_*({}_c\Xi) = {}_c\Xi$$

and because $\mu_{\ell^r}\langle\alpha\rangle \subset \mathbb{G}_m \setminus \mu_c$ by our choice of c, the Kummer map gives a class

$$\mathcal{S}_{c,r}^{\langle\alpha\rangle} := \partial_r({}_c\Xi\mid_{\mu_{\ell^r}\langle\alpha\rangle}) \in H^1(\mu_{\ell^r}\langle\alpha\rangle, \mathbb{Z}/\ell^r\mathbb{Z}(1)). \tag{4.2.2}$$

Here the letter $\mathcal{S}$ is chosen in honour of Soulé. As $[\ell]_*({}_c\Xi) = {}_c\Xi$ we get $[\ell]_*\mathcal{S}_{c,r}^{\langle\alpha\rangle} = \mathcal{S}_{c,r-1}^{\langle\alpha\rangle}$ and as in Formula 3.2.3 we can define:

Definition 4.2.1. Let

$$\mathcal{S}_c^{\langle\alpha\rangle} := \varprojlim_r \mathcal{S}_{c,r}^{\langle\alpha\rangle} \in H^1(S, \Lambda(\mathbb{Z}_\ell(1)\langle\alpha\rangle)(1)).$$

be the inverse limit of the classes defined in Equation (4.2.2).

The k-th moment map $\widetilde{\mathrm{mom}}_\alpha^k : \Lambda(\mathbb{Z}_\ell(1)\langle\alpha\rangle) \to \mathbb{Z}_\ell(k)$ induces a map

$$\mathrm{mom}_\alpha^k : H^1(S, \Lambda(\mathbb{Z}_\ell(1)\langle\alpha\rangle)(1)) \xrightarrow{\widetilde{\mathrm{mom}}_\alpha^k} H^1(S, \mathbb{Z}_\ell(k{+}1)) \xrightarrow{\frac{1}{k!N^k}} H^1(S, \mathbb{Q}_\ell(k{+}1)),$$

where the last map is the one from (2.1.1) divided by $k!N^k$.

Proposition 4.2.2. *Let $k \geq 0$ and $c \in \mathbb{Z}$ with $(c, \ell N) = 1$, then*

$$\mathrm{mom}_\alpha^k(\mathcal{S}_c^{\langle\alpha\rangle}) = \frac{1}{k!N^k}(c^2\widetilde{c}_{k+1}(\alpha) - c^{-k}\widetilde{c}_{k+1}(\alpha^c)).$$

In particular, for $c \equiv 1 \mod N$ one has

$$\mathrm{mom}_\alpha^k(\mathcal{S}_c^{\langle\alpha\rangle}) = \frac{1}{k!N^k}\frac{c^{k+2}-1}{c^k}\widetilde{c}_{k+1}(\alpha).$$

Proof. From the formula

$$\mathcal{S}_{c,r}^{\langle\alpha\rangle} = \partial_r({}_c\Xi) = c^2\partial_r(1-z) - \partial_r(1-z^c)$$

and Proposition 3.1.6 we get

$$\mathrm{mom}_{r,\alpha}^k(\mathcal{S}_{c,r}) = c^2\mathrm{tr}(\partial_r(1-z) \cup \tau_{r,\alpha}^{\otimes k}) - \mathrm{tr}(\partial_r(1-z^c) \cup \tau_{r,\alpha}^{\otimes k}).$$

The first summand is $c^2\widetilde{c}_{k+1,r}(\alpha)$. To treat the second summand consider the c-power map $[c] : \mu_{\ell^r}\langle\alpha\rangle \cong \mu_{\ell^r}\langle\alpha^c\rangle$. Then we have a commutative diagram

$$\begin{array}{ccc}
\mathbb{Z}/\ell^r\mathbb{Z}[\mu_{\ell^r}\langle\alpha\rangle] & \xrightarrow{[c]_!} & \mathbb{Z}/\ell^r\mathbb{Z}[\mu_{\ell^r}\langle\alpha^c\rangle] \\
{\scriptstyle \mathrm{mom}^k_{r,\alpha}}\downarrow & & \downarrow{\scriptstyle \mathrm{mom}^k_{r,\alpha^c}} \\
\mathbb{Z}/\ell^r\mathbb{Z}(k) & \xrightarrow{[c]^k} & \mathbb{Z}/\ell^r\mathbb{Z}(k)
\end{array}$$

We get

$$\begin{array}{ccccc}
H^1(\mu_{\ell^r}\langle\alpha\rangle, \mathbb{Z}/\ell^r\mathbb{Z}(1)) & \xrightarrow{\cong} & H^1(S, \mathbb{Z}/\ell^r\mathbb{Z}[\mu_{\ell^r}\langle\alpha\rangle](1)) & \xrightarrow{\mathrm{mom}^k_{r,\alpha}} & H^1(S, \mathrm{Sym}^k(\mathbb{Z}/\ell^r\mathbb{Z}(1))(1)) \\
\downarrow{\scriptstyle [c]_*} & & \downarrow{\scriptstyle [c]_!} & & \downarrow{\scriptstyle [c]^k} \\
H^1(\mu_{\ell^r}\langle\alpha^c\rangle, \mathbb{Z}/\ell^r\mathbb{Z}(1)) & \xrightarrow{\cong} & H^1(S, \mathbb{Z}/\ell^r\mathbb{Z}[\mu_{\ell^r}\langle\alpha^c\rangle](1)) & \xrightarrow{\mathrm{mom}^k_{r,\alpha^c}} & H^1(S, \mathrm{Sym}^k\mathbb{Z}/\ell^r\mathbb{Z}(1))(1)).
\end{array}$$

As $[c]_*(1-z^c) = (1-z)$, this gives

$$c^k\mathrm{mom}^k_{r,\alpha}(\partial_r(1-z^c)) = \mathrm{mom}^k_{r,\alpha^c}(\partial_r(1-z))$$

and the second summand is $c^{-k}\widetilde{c}_{k+1,r}(\alpha^c)$. □

5. Elliptic Soulé elements

In this section we describe an analogue of the cyclotomic Soulé-Deligne elements on the modular curve.

5.1. **Elliptic and modular units.**

The theory of elliptic units was amplified and simplified by the work of Kato. We recall his main theorem from [Kat04].

Theorem 5.1.1 (Kato [Kat04] 1.10.). *Let $\mathcal{E}$ be an elliptic curve over a scheme S and c be an integer prime to 6, then there exists a unit ${}_c\vartheta_{\mathcal{E}} \in \mathcal{O}(\mathcal{E} \setminus \mathcal{E}[c])^\times$ such that*

1. $\mathrm{div}\,{}_c\vartheta_{\mathcal{E}} = c^2(0) - \mathcal{E}[c]$
2. $[d]_*{}_c\vartheta_{\mathcal{E}} = {}_c\vartheta_{\mathcal{E}}$ *for all d prime to c*
3. *If $\varphi : \mathcal{E} \to \mathcal{E}'$ is an isogeny of elliptic curves over S with $\deg\varphi$ prime to c, then*
$$\varphi_*({}_c\vartheta_{\mathcal{E}}) = {}_c\vartheta_{\mathcal{E}'}.$$
4. *For $\tau \in \mathbb{H}$ and $z \in \mathbb{C} \setminus c^{-1}(\mathbb{Z}\tau + \mathbb{Z})$ let ${}_c\vartheta(\tau, z)$ be the value at z of ${}_c\vartheta_{\mathcal{E}}$ for the elliptic curve $\mathcal{E} = \mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z})$ over $\mathbb{C}$. Then*
$${}_c\vartheta(\tau, z) = q_\tau^{(c^2-1)/12}(-q_z)^{(c-c^2)/2}\frac{(1-q_z)^{c^2}}{1-q_z^c}\widetilde{\gamma}_{q_\tau}(q_z)^{c^2}\widetilde{\gamma}_{q_\tau}(q_z^c)^{-1}$$
where $q_\tau := e^{2\pi i\tau}$, $q_z := e^{2\pi i z}$ and
$$\widetilde{\gamma}_{q_\tau}(t) := \prod_{n\geq 1}(1-q_\tau^n q_z)\prod_{n\geq 1}(1-q_\tau^n q_z^{-1}).$$

Corollary 5.1.2. *Let $t = \frac{a\tau}{N} + \frac{b}{N} \in \mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z})$ be an N-torsion point, $a, b \in \mathbb{Z}$ and let $\zeta_N := e^{\frac{2\pi i}{N}}$, then*

$$ {}_c\vartheta(\tau, t) = q_\tau^{\frac{1}{2}(c^2 B_2(\{\frac{a}{N}\}) - B_2(\{\frac{ca}{N}\}))}(-\zeta_N^b)^{\frac{c-c^2}{2}} \frac{(1 - q_\tau^{\frac{a}{N}}\zeta_N^b)^{c^2}}{(1 - q_\tau^{\frac{ca}{N}}\zeta_N^{cb})} \frac{\widetilde{\gamma}_{q_\tau}(q_\tau^{\frac{a}{N}}\zeta_N^b)^{c^2}}{\widetilde{\gamma}_{q_\tau}(q_\tau^{\frac{ca}{N}}\zeta_N^{cb})} $$

Proof. This follows from Theorem 5.1.1 by writing $q_z = q_\tau^{\frac{a}{N}}\zeta_N^b$ and a straightforward computation using $B_2(x) = x^2 - x + \frac{1}{6}$, so that

$$ c^2 B_2(\{\frac{a}{N}\}) - B_2(\{\frac{ca}{N}\}) = \frac{(c - c^2)a}{N} + \frac{c^2 - 1}{6}. $$

$\square$

5.2. **Elliptic Soulé elements.** We proceed as in Section 4.1. In fact, we start with the norm-compatible functions ${}_c\vartheta_{\mathcal{E}}$, which are the elliptic analogues of the ${}_c\Xi$, and consider their image under the Kummer map. The resulting class is then cupped with a section $\tau_r^{\otimes k} \in H^0(\mathcal{E}[\ell^r]\langle t\rangle, \mathrm{Sym}^k \mathscr{H}_{\mathbb{Z}/\ell^r\mathbb{Z}})$ and as in Section 4.1 one gets elements which form a projective system.

Consider an elliptic curve $\pi : \mathcal{E} \to S$ and $\mathcal{E}[N] \subset \mathcal{E}$. For each section $t \in \mathcal{E}[N](S)$ one has a cartesian diagram

$$ \begin{tikzcd} \mathcal{E}[\ell^r]\langle t\rangle \arrow[r] \arrow[d, "p_r\langle t\rangle"'] & \mathcal{E}[\ell^r N] \arrow[r] \arrow[d] & \mathcal{E} \arrow[d, "{[\ell^r]}"] \\ S \arrow[r, "t"] & \mathcal{E}[N] \arrow[r] & \mathcal{E}. \end{tikzcd} $$

Let $c \in \mathbb{Z}$ with $(c, 6\ell N) = 1$ and consider the function

$$ {}_c\vartheta_{\mathcal{E}} : \mathcal{E} \setminus \mathcal{E}[c] \to \mathbb{G}_m. $$

By Theorem 5.1.1 this function is norm-compatible

$$ [\ell^r]_*({}_c\vartheta_{\mathcal{E}}) = {}_c\vartheta_{\mathcal{E}}. $$

Note that for $t \neq e$ one has $\mathcal{E}[\ell^r]\langle t\rangle \subset \mathcal{E} \setminus \mathcal{E}[c]$, by our condition on c. Thus, we can restrict ${}_c\vartheta_{\mathcal{E}}$ to an invertible function on $\mathcal{E}[\ell^r]\langle t\rangle$. The Kummer map gives a class

$$ \partial_r({}_c\vartheta_{\mathcal{E}}) \in H^1(\mathcal{E}[\ell^r]\langle t\rangle, \mathbb{Z}/\ell^r\mathbb{Z}(1)). $$

As in Definition 3.1.1, we have a section

$$ \tau_{r,t} \in H^0(\mathcal{E}[\ell^r]\langle t\rangle, \mathscr{H}_{\mathbb{Z}/\ell^r\mathbb{Z}}) $$

given by

$$ \tau_{r,t} : \mathcal{E}[\ell^r]\langle t\rangle \hookrightarrow \mathcal{E}[\ell^r N] \xrightarrow{[N]} \mathcal{E}[\ell^r]. \tag{5.2.1} $$

Its k-symmetric power gives

$$ \tau_{r,t}^{\otimes k} \in H^0(\mathcal{E}[\ell^r]\langle t\rangle, \mathrm{Sym}^k \mathscr{H}_{\mathbb{Z}/\ell^r\mathbb{Z}}). $$

If $p_r\langle t\rangle : \mathcal{E}[\ell^r]\langle t\rangle \to S$ is the structure map, we get as in Section 4.1 elements

$$ {}_c\widetilde{e}_{k,r}(t) := p_r\langle t\rangle_*(\partial_r({}_c\vartheta_{\mathcal{E}}) \cup \tau_{r,t}^{\otimes k}) \in H^1(S, \mathrm{Sym}^k \mathscr{H}_{\mathbb{Z}/\ell^r\mathbb{Z}}(1)). \tag{5.2.2} $$

Let $S(\ell^r N) \to S$ be the étale scheme, which represents the level $\ell^r N$-structures on $\mathcal{E}$. Then over $S(\ell^r N)$ the group scheme $\mathcal{E}[\ell^r N]$ is isomorphic to $(\mathbb{Z}/\ell^r N\mathbb{Z})^2$ and one has $\mathscr{H}_{\mathbb{Z}/\ell^r\mathbb{Z}} \cong (\mathbb{Z}/\ell^r\mathbb{Z})^2$. The pull-back of ${}_c\widetilde{e}_{k,r}(t)$ to $S(\ell^r N)$ is given explicitly by

$$ {}_c\widetilde{e}_{k,r}(t) = \sum_{[\ell^r]Q=t} \partial_r({}_c\vartheta_{\mathcal{E}}(Q)) \cup ([N]Q)^{\otimes k} \in H^1(S(\ell^r N), \mathrm{Sym}^k(\mathbb{Z}/\ell^r\mathbb{Z})^2(1)). $$

The same argument as Lemma 4.1.2 shows that the ${}_c\widetilde{e}_{k,r}(t)$ are compatible with respect to reduction modulo ℓ^{r-1}.

Definition 5.2.1. For $t \in \mathcal{E}[N] \setminus \{e\}$ the *elliptic Soulé element* is

$$ {}_c\widetilde{e}_k(t) := \varprojlim_r {}_c\widetilde{e}_{k,r}(t) \in H^1(S, \mathrm{Sym}^k \mathscr{H}_{\mathbb{Z}_\ell}(1)). $$

For a function $\psi : (\mathcal{E}[N](S) \setminus \{e\}) \to \mathbb{Z}_\ell$ we let

$$ {}_c\widetilde{e}_k(\psi) := \sum_{t\in\mathcal{E}[N](S)\setminus\{e\}} \psi(t)\, {}_c\widetilde{e}_k(t). $$

5.3. **Elliptic Soulé elements as moments.** The elliptic Soulé elements can be obtained as moments from a cohomology class

$$ \mathcal{E}\mathcal{S}_c^{\langle t\rangle} \in H^1(S, \Lambda(\mathscr{H}_{\mathbb{Z}_\ell}\langle t\rangle)(1)), $$

which we are going to define: consider with the identification of Lemma 3.2.2

$$ \mathcal{E}\mathcal{S}_{c,r}^{\langle t\rangle} := \partial_r({}_c\vartheta_{\mathcal{E}}) \in H^1(S, \mathbb{Z}/\ell^r\mathbb{Z}[\mathscr{H}_{\mathbb{Z}/\ell^r\mathbb{Z}}\langle t\rangle](1)). \tag{5.3.1} $$

The norm-compatibility $[\ell^r]_*({}_c\vartheta_{\mathcal{E}}) = {}_c\vartheta_{\mathcal{E}}$ allows to define in the limit

$$ \mathcal{E}\mathcal{S}_c^{\langle t\rangle} := \varprojlim_r \mathcal{E}\mathcal{S}_{c,r}^{\langle t\rangle} \in H^1(S, \Lambda(\mathscr{H}_{\mathbb{Z}_\ell}\langle t\rangle)(1)). \tag{5.3.2} $$

The k-th moment map $\widetilde{\mathrm{mom}}_t^k : \Lambda(\mathscr{H}_{\mathbb{Z}_\ell}\langle t\rangle) \to \mathrm{Sym}^k \mathscr{H}_{\mathbb{Z}_\ell}$ induces

$$ \mathrm{mom}_t^k : H^1(S, \Lambda(\mathscr{H}_{\mathbb{Z}_\ell}\langle t\rangle)(1)) \xrightarrow{\widetilde{\mathrm{mom}}_t^k} H^1(S, \mathrm{Sym}^k \mathscr{H}_{\mathbb{Z}_\ell}(1)) \xrightarrow{\frac{1}{k!N^k}} H^1(S, \mathrm{Sym}^k \mathscr{H}_{\mathbb{Q}_\ell}(1)), $$

where the last map is (2.1.1) divided by $k!N^k$.

Proposition 5.3.1. *Let $k \geq 0$ and $c \in \mathbb{Z}$ with $(c, 6\ell N) = 1$. Then*

$$ \mathrm{mom}_t^k(\mathcal{E}\mathcal{S}_c^{\langle t\rangle}) = \frac{1}{N^k k!}\, {}_c\widetilde{e}_k(t). $$

Proof. From Proposition 3.1.6 and the definition of ${}_c\widetilde{e}_k(t)$ one gets that

$$ \mathrm{mom}_{r,t}^k : H^1(S, \mathbb{Z}/\ell^r\mathbb{Z}[\mathscr{H}_{\mathbb{Z}/\ell^r\mathbb{Z}}\langle t\rangle](1)) \to H^1(S, \mathrm{Sym}^k \mathscr{H}_{\mathbb{Z}/\ell^r\mathbb{Z}}(1)) $$

maps $\mathcal{E}\mathcal{S}_{c,r}^{\langle t\rangle}$ to ${}_c\widetilde{e}_{k,r}(t)$ and the proposition follows by taking the projective limit. $\square$

6. Modular curves

6.1. **Some facts on modular curves.** We fix some notation concerning modular curves and list the facts we need later.

Definition 6.1.1. Let $N \geq 3$ and denote by $Y(N)/\mathrm{Spec}\mathbb{Q}$ the moduli scheme of elliptic curves with a level-N-structure, i.e., for a scheme S we have

$$Y(N)(S) = \{(\mathcal{E}/S, \alpha) \mid \mathcal{E}/S \text{ elliptic curve }, \alpha : (\mathbb{Z}/N\mathbb{Z})^2 \cong \mathcal{E}[N]\}.$$

The universal elliptic curve is denoted by

$$\pi : \mathcal{E} \to Y(N)$$

and we let $X(N)$ be the smooth compactification of $Y(N)$. The scheme

$$\mathrm{Cusp} := X(N) \setminus Y(N)$$

is called the scheme of cusps.

We recall some standard facts about $Y(N)$. First note that $\sigma \in \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ acts from the left on $Y(N)$ by $\sigma\alpha(v) := \alpha(v\sigma)$ for all $v \in (\mathbb{Z}/N\mathbb{Z})^2$. Let $\mathbb{H} := \{\tau \in \mathbb{C} \mid \mathrm{Im}\tau > 0\}$ be the upper half plane, then one has an analytic uniformization

$$\begin{aligned} \nu : (\mathbb{Z}/N\mathbb{Z})^\times \times (\Gamma(N)\backslash\mathbb{H}) &\xrightarrow{\cong} Y(N)(\mathbb{C}) \\ (a, \tau) &\mapsto (\mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z}), \alpha), \end{aligned} \tag{6.1.1}$$

where $\Gamma(N) := \ker(\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}))$ and α is the level structure given by $(v_1, v_2) \mapsto \frac{av_1\tau + v_2}{N}$. Let $\zeta_N = e^{2\pi i/N} \in \mathbb{C}$ and consider over $\mathrm{Spec}\mathbb{Q}(\zeta_N)((q^{1/N}))$ the Tate curve $\mathcal{E}_q$ with the level structure $\alpha : (\mathbb{Z}/N\mathbb{Z})^2 \to \mathcal{E}_q[N]$ given by $(a, b) \mapsto q^a\zeta_N^b$. The corresponding map of schemes

$$\mathrm{Spec}\mathbb{Q}(\zeta_N)((q^{1/N})) \to Y(N)$$

induces $\mathrm{Spec}\mathbb{Q}(\zeta_N)[[q^{1/N}]] \to X(N)$ and a hence a map $\infty : \mathrm{Spec}\mathbb{Q}(\zeta_N) \to \mathrm{Cusp}$, whose image we call the cusp ∞.

The cusps are permuted transitively by the $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$-action and the cusp ∞ has stabilizer $\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$. The scheme of cusps has the form

$$\mathrm{Cusp} = \coprod_{\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\pm\left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right)} \mathrm{Spec}\mathbb{Q}(\mu_N). \tag{6.1.2}$$

Let $\widehat{X}(N)_\infty$ be the completion of $X(N)$ at ∞, which can be identified via the above map with $\mathrm{Spec}\mathbb{Q}(\zeta_N)[[q^{1/N}]]$. We denote by $\widehat{Y}(N)_\infty$ the generic fibre of $\widehat{X}(N)_\infty$ so that $\widehat{Y}(N)_\infty \cong \mathrm{Spec}\mathbb{Q}(\zeta_N)((q^{1/N}))$. One has a commutative diagram

$$\begin{array}{ccccc} \widehat{Y}(N)_\infty & \xrightarrow{j} & \widehat{X}(N)_\infty & \xleftarrow{\infty} & \infty \\ \downarrow & & \downarrow & & \downarrow{=} \\ Y(N) & \xrightarrow{j} & Y(N) \amalg \infty & \xleftarrow{\infty} & \infty. \end{array} \tag{6.1.3}$$

Consider the Tate curve $\mathcal{E}_q$ over $\widehat{Y}(N)_\infty$. For each $M > 1$ one has an exact sequence

$$0 \to \mu_M \xrightarrow{\iota} \mathcal{E}_q[M] \xrightarrow{p} \mathbb{Z}/M\mathbb{Z} \to 0 \tag{6.1.4}$$

or correspondingly for the associated étale sheaves

$$0 \to \mathbb{Z}/M\mathbb{Z}(1) \xrightarrow{\iota} \mathscr{H}_{\mathbb{Z}/M\mathbb{Z}} \xrightarrow{p} \mathbb{Z}/M\mathbb{Z} \to 0. \tag{6.1.5}$$

As a special case we get in the limit

$$0 \to \mathbb{Q}_\ell(1) \xrightarrow{\iota} \mathscr{H}_{\mathbb{Q}_\ell} \xrightarrow{p} \mathbb{Q}_\ell \to 0. \tag{6.1.6}$$

Proposition 6.1.2. *The subsheaf $\iota(\mathbb{Z}/M\mathbb{Z}(1)) \subset \mathscr{H}_{\mathbb{Z}/M\mathbb{Z}}$ are the invariants of monodromy, i.e.,*

$$\infty^* j_* \mathscr{H}_{\mathbb{Z}/M\mathbb{Z}} \cong \mathbb{Z}/M\mathbb{Z}(1) \qquad \textit{and} \qquad \infty^* R^1 j_* \mathscr{H}_{\mathbb{Z}/M\mathbb{Z}} \cong \mathbb{Z}/M\mathbb{Z}(-1),$$

where the first isomorphism is induced by ι and the second isomorphism is the composition

$$\infty^* R^1 j_* \mathscr{H}_{\mathbb{Z}/M\mathbb{Z}} \xrightarrow{p} \infty^* R^1 j_* \mathbb{Z}/M\mathbb{Z} \cong \mathbb{Z}/M\mathbb{Z}(-1).$$

Proof. That $\iota(\mathbb{Z}/M\mathbb{Z}(1)) \subset \mathscr{H}_{\mathbb{Z}/M\mathbb{Z}}$ are the invariants of monodromy is [SGA72, Exposé IX, Proposition 2.2.5 and (2.2.5.1)]. From the long exact sequence for $\infty^* Rj_*$ we get

$$0 \to \infty^* j_* \mathbb{Z}/M\mathbb{Z} \to \infty^* R^1 j_* \mathbb{Z}/M\mathbb{Z}(1) \to \infty^* R^1 j_* \mathscr{H}_{\mathbb{Z}/M\mathbb{Z}} \to \infty^* R^1 j_* \mathbb{Z}/M\mathbb{Z} \to 0.$$

As $\infty^* j_* \mathbb{Z}/M\mathbb{Z} \cong \mathbb{Z}/M\mathbb{Z}$ and $\mathbb{Z}/M\mathbb{Z} \cong \infty^* R^1 j_* \mathbb{Z}/M\mathbb{Z}(1)$, the first map is an isomorphism and one gets $\infty^* R^1 j_* \mathscr{H}_{\mathbb{Z}/M\mathbb{Z}} \cong \mathbb{Z}/M\mathbb{Z}(-1)$. □

Corollary 6.1.3. *Over $\widehat{Y}(N)_\infty$ the maps induced by ι and p*

$$\mathbb{Q}_\ell(k) \cong \mathrm{Sym}^k \mathbb{Q}_\ell(1) \xrightarrow{\iota} \mathrm{Sym}^k \mathscr{H}_{\mathbb{Q}_\ell}$$

$$\mathrm{Sym}^k \mathscr{H}_{\mathbb{Q}_\ell} \xrightarrow{p} \mathrm{Sym}^k \mathbb{Q}_\ell \cong \mathbb{Q}_\ell$$

induce isomorphisms

$$\infty^* j_* \mathrm{Sym}^k \mathscr{H}_{\mathbb{Q}_\ell} \cong \mathbb{Q}_\ell(k) \qquad \textit{and} \qquad \infty^* R^1 j_* \mathrm{Sym}^k \mathscr{H}_{\mathbb{Q}_\ell} \cong \mathbb{Q}_\ell(-1).$$

Proof. This follows by induction on k from Proposition 6.1.2 and the exact sequence

$$0 \to \mathbb{Q}_\ell(k) \xrightarrow{\iota} \mathrm{Sym}^k \mathscr{H}_{\mathbb{Q}_\ell} \to \mathrm{Sym}^{k-1} \mathscr{H}_{\mathbb{Q}_\ell} \to 0.$$

□

Definition 6.1.4. Let $\widetilde{Y}(N) := Y(N) \amalg \infty$, then the *residue* at ∞ is the map

$$\mathrm{res}_\infty : H^1(\widetilde{Y}(N), Rj_* \mathrm{Sym}^k \mathscr{H}_{\mathbb{Q}_\ell}(1)) \to H^0(\widetilde{Y}(N), R^1 j_* \mathrm{Sym}^k \mathscr{H}_{\mathbb{Q}_\ell}(1)) \cong H^0(\infty, \mathbb{Q}_\ell),$$

induced from the edge morphism of the Leray spectral sequence for Rj_*.

Note that by base change one also gets a residue map

$$\mathrm{res}_\infty : H^1(\widehat{Y}(N)_\infty, \mathrm{Sym}^k \mathscr{H}_{\mathbb{Q}_\ell}(1)) \to H^0(\infty, \mathbb{Q}_\ell). \tag{6.1.7}$$

Corollary 6.1.5. *Let $\widetilde{Y}(N) := Y(N) \amalg \infty$, then one has a commutative diagram of exact sequences*

$$\begin{tikzcd}
0 \arrow[r] & H^1(\widetilde{Y}(N), j_*\mathrm{Sym}^k\mathscr{H}_{\mathbb{Q}_\ell}(1)) \arrow[r] \arrow[d] & H^1(Y(N), \mathrm{Sym}^k\mathscr{H}_{\mathbb{Q}_\ell}(1)) \arrow[r, "\mathrm{res}_\infty"] \arrow[d] & H^0(\infty, \mathbb{Q}_\ell) \arrow[d, equal] \\
0 \arrow[r] & H^1(\widehat{X}(N)_\infty, j_*\mathrm{Sym}^k\mathscr{H}_{\mathbb{Q}_\ell}(1)) \arrow[r] & H^1(\widehat{Y}(N)_\infty, \mathrm{Sym}^k\mathscr{H}_{\mathbb{Q}_\ell}(1)) \arrow[r, "\mathrm{res}_\infty"] & H^0(\infty, \mathbb{Q}_\ell)
\end{tikzcd}$$

where res_∞ *is the residue defined in 6.1.4.*

Proof. The exact sequences follow from the Leray spectral sequence for Rj_* and the above computations. Observe that $R^1j_*\mathrm{Sym}^k\mathscr{H}_{\mathbb{Q}_\ell}$ is supported on ∞. The commutativity of the diagram follows from diagram (6.1.3). □

7. The residue at ∞ of the elliptic Soulé elements

7.1. **A residue theorem at finite level.** In this section we write

$$\widehat{T} := \widehat{Y}(N)_\infty = \mathrm{Spec}\mathbb{Q}(\zeta_N)[[q^{1/N}]]$$

and we consider the map $\widehat{T} \to Y(N)$ induced by the Tate curve and its canonical level structure $(a, b) \mapsto q^a\zeta_N^b$. The pull-back of the ℓ^rN-torsion points $\mathcal{E}[\ell^rN]$ of the universal elliptic curve $\pi : \mathcal{E} \to Y(N)$ to $\widehat{T}$ sits in an extension of finite étale group schemes

$$\begin{tikzcd}
0 \arrow[r] & \mu_{\ell^rN,\widehat{T}} \arrow[r] & \mathcal{E}[\ell^rN]_{\widehat{T}} \arrow[r, "p"] \arrow[d, "{[\ell^r]}"] & \mathbb{Z}/\ell^rN\mathbb{Z}_{\widehat{T}} \arrow[r] \arrow[d, "{[\ell^r]}"] & 0 \\
& & \mathcal{E}[N]_{\widehat{T}} \arrow[r, "p"] & \mathbb{Z}/N\mathbb{Z}_{\widehat{T}} &
\end{tikzcd} \tag{7.1.1}$$

For each section $t : \widehat{T} \to \mathcal{E}[N]$ the pull-back by t induces a finite morphism

$$p : \mathcal{E}[\ell^r]\langle t\rangle \to \mathbb{Z}/\ell^r\langle p(t)\rangle$$

Definition 7.1.1. Let $\mathrm{res}_{r,\infty}$ be the following composition of maps:

$$\begin{aligned}
\mathrm{res}_{r,\infty} : H^1(\mathcal{E}[\ell^r]\langle t\rangle, \mathbb{Z}/\ell^r\mathbb{Z}(1)) \to H^1(\mathcal{E}[\ell^r]\langle t\rangle_{\widehat{T}}, \mathbb{Z}/\ell^r\mathbb{Z}(1)) \xrightarrow{p_*} \\
\xrightarrow{p_*} H^1(\mathbb{Z}/\ell^r\mathbb{Z}\langle p(t)\rangle_{\widehat{T}}, \mathbb{Z}/\ell^r\mathbb{Z}(1)) \xrightarrow{\mathrm{res}_\infty} H^0(\mathbb{Z}/\ell^r\mathbb{Z}\langle p(t)\rangle_\infty, \mathbb{Z}/\ell^r\mathbb{Z}),
\end{aligned}$$

where the first map is the pull-back to $\widehat{T}$. With the identifications in Lemma 3.2.2 we define the map

$$\mathrm{res}_\infty : H^1(Y(N), \Lambda(\mathscr{H}_{\mathbb{Z}_\ell}\langle t\rangle)(1)) \to H^0(\infty, \Lambda(\mathbb{Z}_\ell\langle p(t)\rangle)) \cong \Lambda(\mathbb{Z}_\ell\langle p(t)\rangle)$$

to be the inverse limit $\mathrm{res}_\infty := \varprojlim_r \mathrm{res}_{r,\infty}$.

The residue of the elliptic Soulé elements will be deduced from the following fundamental theorem:

Theorem 7.1.2. *Let*

$$\mathcal{E}\mathcal{S}_{c,r}^{\langle t\rangle} = \partial_r({}_c\vartheta_{\mathcal{E}}) \in H^1(\mathcal{E}[\ell^r]\langle t\rangle, \mathbb{Z}/\ell^r\mathbb{Z}(1))$$

be the element defined in 5.3.1, then

$$\mathrm{res}_{r,\infty}(\mathcal{E}\mathcal{S}_{c,r}^{\langle t\rangle}) = B_{2,c,r}^{\langle p(t)\rangle},$$

where

$$B_{2,c,r}^{\langle p(t)\rangle} \in H^0(\mathbb{Z}/\ell^r\mathbb{Z}\langle p(t)\rangle_\infty, \mathbb{Z}/\ell^r\mathbb{Z}) \cong \mathbb{Z}/\ell^r\mathbb{Z}[\mathbb{Z}/\ell^r\mathbb{Z}\langle p(t)\rangle]$$

is the element defined in Formula (1.1.7)*.*

Taking the projective limit we get:

Corollary 7.1.3. *Let $\mathcal{E}\mathcal{S}_c^{\langle t\rangle} \in H^1(Y(N), \Lambda(\mathscr{H}_{\mathbb{Z}_\ell}\langle t\rangle)(1))$ be the element defined in Equation* (5.3.2)*. Then*

$$\mathrm{res}_\infty(\mathcal{E}\mathcal{S}_c^{\langle t\rangle}) = B_{2,c}^{\langle p(t)\rangle}$$

where $B_{2,c}^{\langle p(t)\rangle} \in H^0(\infty, \Lambda(\mathbb{Z}_\ell\langle p(t)\rangle)) \cong \Lambda(\mathbb{Z}_\ell\langle p(t)\rangle)$ is the Bernoulli measure defined in Equation (1.1.8)*.*

The proof of Theorem 7.1.2 will occupy the rest of this section.

Let us write $\mathbb{Z}[\mathbb{Z}/\ell^r\mathbb{Z}\langle p(t)\rangle]$ for the $\mathbb{Z}$-valued functions on $\mathbb{Z}/\ell^r\mathbb{Z}\langle p(t)\rangle$ and note that ord_∞ induces a map

$$\mathbb{G}_m(\mathbb{Z}/\ell^r\mathbb{Z}\langle p(t)\rangle_{\widehat{T}_{\ell^r N}}) \cong \prod_{x\in\mathbb{Z}/\ell^r\mathbb{Z}\langle p(t)\rangle} \mathbb{G}_m(\widehat{T}_{\ell^r N}) \xrightarrow{\mathrm{ord}_\infty} \prod_{x\in\mathbb{Z}/\ell^r\mathbb{Z}\langle p(t)\rangle} \mathbb{Z} = \mathbb{Z}[\mathbb{Z}/\ell^r\mathbb{Z}\langle p(t)\rangle].$$

We start with a compatibility:

Lemma 7.1.4. *Identify*

$$H^0(\mathbb{Z}/\ell^r\mathbb{Z}\langle p(t)\rangle_\infty, \mathbb{Z}/\ell^r\mathbb{Z}) \cong \mathbb{Z}/\ell^r\mathbb{Z}[\mathbb{Z}/\ell^r\mathbb{Z}\langle p(t)\rangle].$$

Then following diagram commutes:

$$\begin{tikzcd}
\mathbb{G}_m(\mathcal{E}[\ell^r]\langle t\rangle_{\widehat{T}}) \arrow[r, "p_*"] \arrow[d, "\partial_r"] & \mathbb{G}_m(\mathbb{Z}/\ell^r\mathbb{Z}\langle p(t)\rangle_{\widehat{T}}) \arrow[r, "\mathrm{ord}_\infty"] \arrow[d, "\partial_r"] & \mathbb{Z}[\mathbb{Z}/\ell^r\mathbb{Z}\langle p(t)\rangle] \arrow[d] \\
H^1(\mathcal{E}[\ell^r]\langle t\rangle_{\widehat{T}}, \mathbb{Z}/\ell^r\mathbb{Z}(1)) \arrow[r, "p_*"] & H^1(\mathbb{Z}/\ell^r\mathbb{Z}\langle p(t)\rangle_{\widehat{T}}, \mathbb{Z}/\ell^r\mathbb{Z}(1)) \arrow[r, "\mathrm{res}_\infty"] & \mathbb{Z}/\ell^r\mathbb{Z}[\mathbb{Z}/\ell^r\mathbb{Z}\langle p(t)\rangle]
\end{tikzcd}$$

Here the upper horizontal p_* is the norm with respect to p and the right vertical arrow is reduction modulo ℓ^r.

Proof. Compatibility of the Kummer map with traces and residues. □

From this diagram it follows that to show Theorem 7.1.2 it suffices to compute $\mathrm{ord}_\infty \circ p_{*c}\vartheta_{\mathcal{E}}$. To calculate this, we write $\widehat{T}_N := \widehat{Y}(N)_\infty = \mathrm{Spec}\mathbb{Q}(\zeta_N)((q^{1/N}))$ and $\widehat{T}_{\ell^r N} := \widehat{Y}(\ell^r N)_\infty = \mathrm{Spec}\mathbb{Q}(\zeta_{\ell^r N})((q^{1/\ell^r N}))$ and perform a base change to $\widehat{T}_{\ell^r N}$. Over $\widehat{T}_{\ell^r N}$ the scheme $\mathcal{E}[\ell^r]\langle t\rangle$ is isomorphic to

$$(\mathbb{Z}/\ell^r\mathbb{Z})^2\langle t\rangle := \{(x,y) \in (\mathbb{Z}/\ell^r N)^2 \mid [\ell^r](x,y) = t\}$$

and the map $p:(\mathbb{Z}/\ell^r\mathbb{Z})^2\langle t\rangle_{\widehat{T}_{\ell^r N}}\to\mathbb{Z}/\ell^r\mathbb{Z}\langle p(t)\rangle_{\widehat{T}_{\ell^r N}}$ is given by the projection onto the first coordinate $(x,y)\mapsto x$.

Lemma 7.1.5. *One has a commutative diagram*

$$\begin{tikzcd}
\mathbb{G}_m(\mathcal{E}[\ell^r]\langle t\rangle_{\widehat{T}_N}) \arrow[r] \arrow[d, "p_*"'] & \mathbb{G}_m((\mathbb{Z}/\ell^r\mathbb{Z})^2\langle t\rangle_{\widehat{T}_{\ell^r N}}) \arrow[d, "p_*"] \\
\mathbb{G}_m(\mathbb{Z}/\ell^r\mathbb{Z}\langle p(t)\rangle_{\widehat{T}_N}) \arrow[r] \arrow[d, "\mathrm{ord}_\infty"'] & \mathbb{G}_m(\mathbb{Z}/\ell^r\mathbb{Z}\langle p(t)\rangle_{\widehat{T}_{\ell^r N}}) \arrow[d, "\mathrm{ord}_\infty"] \\
\mathbb{Z}[\mathbb{Z}/\ell^r\mathbb{Z}\langle p(t)\rangle] \arrow[r, "\ell^r"] & \mathbb{Z}[\mathbb{Z}/\ell^r\mathbb{Z}\langle p(t)\rangle],
\end{tikzcd}$$

where the lower horizontal map is multiplication by ℓ^r.

Proof. As the map $\widehat{T}_{\ell^r N}\to\widehat{T}_N$ is ramified of degree ℓ^r one has to multiply the order with the ramification degree. $\square$

Furthermore:

Lemma 7.1.6. *The diagram*

$$\begin{tikzcd}
\mathbb{G}_m((\mathbb{Z}/\ell^r\mathbb{Z})^2\langle t\rangle_{\widehat{T}_{\ell^r N}}) \arrow[r, "\mathrm{ord}_\infty"] \arrow[d, "p_*"] & \mathbb{Z}[(\mathbb{Z}/\ell^r\mathbb{Z})^2\langle t\rangle] \arrow[d, "p_!"] \\
\mathbb{G}_m(\mathbb{Z}/\ell^r\mathbb{Z}\langle p(t)\rangle_{\widehat{T}_{\ell^r N}}) \arrow[r, "\mathrm{ord}_\infty"] & \mathbb{Z}[\mathbb{Z}/\ell^r\mathbb{Z}\langle p(t)\rangle]
\end{tikzcd}$$

commutes, where $p_!$ is the trace map associated to p.

Proof. This is a direct consequence from the fact that $\mathrm{div}(p_*(f))=p_*\mathrm{div}(f)$ for any function $f\in\mathbb{G}_m((\mathbb{Z}/\ell^r\mathbb{Z})^2\langle t\rangle_{\widehat{T}_{\ell^r N}})$. $\square$

It follows that to compute $\mathrm{ord}_\infty\circ p_{*c}\vartheta_{\mathcal{E}}$ we can first pull-back the function ${}_c\vartheta_{\mathcal{E}}$ to $\mathbb{G}_m((\mathbb{Z}/\ell^r\mathbb{Z})^2\langle t\rangle_{\widehat{T}_{\ell^r N}})$ and then take $\frac{1}{\ell^r}p_!\mathrm{ord}_\infty$. Note that

$$\mathbb{G}_m((\mathbb{Z}/\ell^r\mathbb{Z})^2\langle t\rangle_{\widehat{T}_{\ell^r N}})\cong\prod_{(x,y)\in(\mathbb{Z}/\ell^r\mathbb{Z})^2\langle t\rangle}\mathbb{G}_m(\widehat{T}_{\ell^r N})$$

and the (x,y)-component of the pull-back of ${}_c\vartheta_{\mathcal{E}}$ is

$$(x,y)^*{}_c\vartheta_{\mathcal{E}}\in\mathbb{G}_m(\widehat{T}_{\ell^r N}).$$

We have to calculate the order of $(x,y)^*{}_c\vartheta_{\mathcal{E}}$ at ∞. For this we can work on $Y(\ell^r N)(\mathbb{C})$. From Corollary 5.1.2 we see that $(x,y)^*{}_c\vartheta_{\mathcal{E}}$ is given by

$$q_\tau^{\frac{1}{2}(c^2B_2(\{\frac{x}{\ell^r N}\})-B_2(\{\frac{cx}{\ell^r N}\}))}(-\zeta_{\ell^r N}^y)^{\frac{c-c^2}{2}}\frac{(1-q_\tau^{\frac{x}{\ell^r N}}\zeta_{\ell^r N}^y)^{c^2}}{(1-q_\tau^{\frac{cx}{\ell^r N}}\zeta_{\ell^r N}^{cy})}\frac{\widetilde{\gamma}_{q_\tau}(q_\tau^{\frac{x}{\ell^r N}}\zeta_{\ell^r N}^y)^{c^2}}{\widetilde{\gamma}_{q_\tau}(q_\tau^{\frac{cx}{\ell^r N}}\zeta_{\ell^r N}^{cy})}.$$

If we observe that the uniformizing parameter for $Y(\ell^r N)(\mathbb{C})$ at ∞ is $q_\tau^{1/\ell^r N}$, we get

$$\operatorname{ord}_\infty((x,y)^*{}_c\vartheta_{\mathcal{E}}) = \frac{\ell^r N}{2}(c^2 B_2(\{\frac{x}{\ell^r N}\}) - B_2(\{\frac{cx}{\ell^r N}\}))$$

As there are ℓ^r elements $(x,y) \in (\mathbb{Z}/\ell^r\mathbb{Z})^2\langle t\rangle$ with $p(x,y) = x \in \mathbb{Z}/\ell^r\mathbb{Z}\langle p(t)\rangle$ we finally find

$$\begin{aligned}\frac{1}{\ell^r} p_! \operatorname{ord}_\infty(\prod_y (x,y)^*{}_c\vartheta_{\mathcal{E}}) &= \frac{\ell^r N}{2}(c^2 B_2(\{\frac{x}{\ell^r N}\}) - B_2(\{\frac{cx}{\ell^r N}\})) \\ &= B_{2,c,r}^{\langle p(t)\rangle}(x).\end{aligned}$$

This finishes the proof of Theorem 7.1.2.

7.2. **The residue at ∞ of the elliptic Soulé elements.** From Theorem 7.1.2 we will deduce in this section a formula for the residue of the elliptic Soulé elements.

Let $\pi : \mathcal{E} \to Y(N)$ be the universal elliptic curve. We use the level structure $\mathcal{E}[N] \cong (\mathbb{Z}/N\mathbb{Z})^2$ to identify N-torsion sections t with elements $(a,b) \in (\mathbb{Z}/N\mathbb{Z})^2$. Note that under the map $p : \mathcal{E}[N] \to \mathbb{Z}/N\mathbb{Z}$ over $\widehat{Y}(N)_\infty$ one has $p(a,b) = a$. Recall that we have defined in 5.2.1 the elliptic Soulé element

$${}_c\widetilde{e}_k(t) \in H^1(Y(N), \operatorname{Sym}^k \mathscr{H}_{\mathbb{Z}_\ell}(1)),$$

which we consider in $H^1(Y(N), \operatorname{Sym}^k \mathscr{H}_{\mathbb{Q}_\ell}(1))$ and in 6.1.4 the residue map

$$\operatorname{res}_\infty : H^1(Y(N), \operatorname{Sym}^k \mathscr{H}_{\mathbb{Q}_\ell}(1)) \to H^0(\infty, \mathbb{Q}_\ell).$$

Theorem 7.2.1. *Let $t = (a,b) \in \mathcal{E}[N](Y(N))$ be as above, then*

$$\operatorname{res}_\infty({}_c\widetilde{e}_k(t)) = \frac{N^{k+1}}{k+2}(c^2 B_{k+2}(\{\frac{a}{N}\}) - c^{-k} B_{k+2}(\{\frac{ca}{N}\}))$$

In particular, if $c \equiv 1 \mod N$ one gets

$$\operatorname{res}_\infty({}_c\widetilde{e}_k(t)) = \frac{N^{k+1}}{k+2}\frac{c^{k+2}-1}{c^k} B_{k+2}(\{\frac{a}{N}\}).$$

To deduce this result from Theorem 7.1.2, we need to compare the different residue maps.

Lemma 7.2.2. *There is a commutative diagram*

$$\begin{tikzcd}
H^1(Y(N), \Lambda(\mathscr{H}_{\mathbb{Z}_\ell}\langle t\rangle)(1)) \arrow[r, "\operatorname{res}_\infty"] \arrow[d, "\widetilde{\operatorname{mom}}_t^k"'] & H^0(\infty, \Lambda(\mathbb{Z}_\ell\langle p(t)\rangle)) \arrow[d, "\widetilde{\operatorname{mom}}_{p(t)}^k"] \\
H^1(Y(N), \operatorname{Sym}^k \mathscr{H}_{\mathbb{Z}_\ell}(1))) \arrow[d] & H^0(\infty, \mathbb{Z}_\ell) \arrow[d] \\
H^1(Y(N), \operatorname{Sym}^k \mathscr{H}_{\mathbb{Q}_\ell}(1))) \arrow[r, "\operatorname{res}_\infty"] & H^0(\infty, \mathbb{Q}_\ell),
\end{tikzcd}$$

where the upper res_∞ *is the one from Definition 7.1.1 and the lower* res_∞ *is defined in 6.1.4.*

Proof. The functoriality of the moment map gives

$$\begin{array}{ccc} \Lambda(\mathscr{H}_{\mathbb{Z}_\ell}\langle t\rangle) & \xrightarrow{p_!} & \Lambda(\mathbb{Z}_\ell\langle p(t)\rangle) \\ {\scriptstyle\widetilde{\mathrm{mom}}_t^k}\downarrow & & \downarrow{\scriptstyle\widetilde{\mathrm{mom}}_{p(t)}^k} \\ \mathrm{Sym}^k\mathscr{H}_{\mathbb{Z}_\ell} & \xrightarrow{\mathrm{Sym}^k p} & \mathbb{Z}_\ell \end{array}$$

and the lemma follows from the definitions. □

Proof of Theorem 7.2.1. By Proposition 5.3.1, Lemma 7.2.2 and the formula in Corollary 7.1.3 one has

$$\begin{aligned} \mathrm{res}_\infty({}_c\widetilde{e}_k(t)) &= \mathrm{res}_\infty(k!N^k\mathrm{mom}_t^k(\mathcal{E}\mathcal{S}_c^{\langle t\rangle})) \\ &= \mathrm{res}_\infty(\widetilde{\mathrm{mom}}_t^k(\mathcal{E}\mathcal{S}_c^{\langle t\rangle})) \\ &= \widetilde{\mathrm{mom}}_t^k(\mathrm{res}_\infty(\mathcal{E}\mathcal{S}_c^{\langle t\rangle})) \\ &= \widetilde{\mathrm{mom}}_t^k(B_{2,c}^{\langle a\rangle}) \\ &= \frac{N^{k+1}}{c^k(k+2)}(c^{k+2}B_{k+2}(\{\frac{a}{N}\}) - B_{k+2}(\{\frac{ca}{N}\})), \end{aligned}$$

where the last equality is Formula 1.2.8. □

8. The integral ℓ-adic elliptic polylogarithm

In this section we define an integral ℓ-adic analogue of the polylogarithm extension.

8.1. **A brief review of the elliptic logarithm.** We give a brief review of the elliptic polylogarithm and refer for more details to the appendix A in [HK99], to [Bla] or the original source [BL94].

Let $\pi : \mathcal{E} \to S$ be a family of elliptic curves with unit section $e : S \to \mathcal{E}$. We consider étale sheaves of F-modules, where $F = \mathbb{Z}/\ell^r\mathbb{Z}, \mathbb{Z}_\ell, \mathbb{Q}_\ell$. A sheaf $\mathscr{G}$ is unipotent of length n with respect to π, if it has a filtration $\mathscr{G} = A^0\mathscr{G} \supset A^{\mathscr{G}} \supset \ldots \supset A^{n+1}\mathscr{G} = 0$ such that $A^k\mathscr{G}/A^{k+1}\mathscr{G} \cong \pi^*\mathscr{F}$ for a lisse sheaf $\mathscr{F}$ on S. Let

$$\mathscr{H}_F := R^1\pi_*F(1) \tag{8.1.1}$$

so that $\mathscr{H}_F$ is one of the sheaves $\mathscr{H}_{\mathbb{Z}/\ell^r\mathbb{Z}}, \mathscr{H}_{\mathbb{Z}_\ell}, \mathscr{H}_{\mathbb{Q}_\ell}$ defined earlier. Let $\mathscr{H}_F^\vee := \underline{\mathrm{Hom}}_S(\mathscr{H}_F, F)$ be the F-dual. The $R\pi_*$ boundary map for the exact sequence

$$0 \to \mathrm{gr}_A^{k+1}\mathscr{G} \to A^k\mathscr{G}/A^{k+2}\mathscr{G} \to \mathrm{gr}_A^k\mathscr{G} \to 0$$

induce $\alpha : \pi_*\mathrm{gr}_A^k\mathscr{G} \xrightarrow{\alpha} \mathscr{H}_F^\vee \otimes \pi_*\mathrm{gr}_A^{k+1}\mathscr{G}$, from which we deduce $\mathscr{H}_F \otimes \pi_*\mathrm{gr}_A^k\mathscr{G} \to \pi_*\mathrm{gr}_A^{k+1}\mathscr{G}$. Thus, $\pi_*\mathrm{gr}_A^{\cdot}\mathscr{G}$ is an $\mathrm{Sym}^{\cdot}\mathscr{H}_F$-module.

Let $\mathscr{G}^{(n)}$ be a unipotent sheaf of length n and suppose that we have a section $1^{(n)} \in \mathrm{Hom}_S(F, e^*\mathscr{G}^{(n)})$, then there is a unique morphism of $\mathrm{Sym}^\cdot\mathscr{H}_F$-modules

$$\nu^\cdot : \mathrm{Sym}^\cdot\mathscr{H}_F/\mathrm{Sym}^{\geq n+1}\mathscr{H}_F \to \pi_*\mathrm{gr}^\cdot_A\mathscr{G}^{(n)} \tag{8.1.2}$$

that maps 1 to $1^{(n)}$ modulo A^1.

Definition 8.1.1. The n-th elliptic logarithm sheaf $\mathscr{L}\mathrm{og}_F^{(n)}$ is the unique unipotent sheaf of length n together with $1^{(n)} \in \mathrm{Hom}_S(F, e^*\mathscr{L}\mathrm{og}_F^{(n)})$ such that $\nu^\cdot$ is an isomorphism.

The uniqueness is [BL94, Proposition 1.2.6]. As $A^n\mathscr{L}\mathrm{og}_F^{(n)} = \pi^*\mathrm{Sym}^n\mathscr{H}_F$ one has an exact sequence

$$0 \to \pi^*\mathrm{Sym}^n\mathscr{H}_F \to \mathscr{L}\mathrm{og}_F^{(n)} \to \mathscr{L}\mathrm{og}_F^{(n-1)} \to 0. \tag{8.1.3}$$

If $F = \mathbb{Q}_\ell$ the logarithm sheaf can be described as follows: $\mathscr{L}\mathrm{og}_{\mathbb{Q}_\ell}^{(1)}$ is the unique extension

$$0 \to \pi^*\mathscr{H}_{\mathbb{Q}_\ell} \to \mathscr{L}\mathrm{og}_{\mathbb{Q}_\ell}^{(1)} \to \mathbb{Q}_\ell \to 0$$

such that the boundary map for $R\pi_*$

$$\mathbb{Q}_\ell \to \mathscr{H}_{\mathbb{Q}_\ell}^\vee \otimes \mathscr{H}_{\mathbb{Q}_\ell} \cong \underline{\mathrm{Hom}}(\mathscr{H}_{\mathbb{Q}_\ell}, \mathscr{H}_{\mathbb{Q}_\ell})$$

maps 1 to id and $e^*\mathscr{L}\mathrm{og}_{\mathbb{Q}_\ell}^{(1)}$ is split by a section $1^{(1)} : \mathbb{Q}_\ell \to e^*\mathscr{L}\mathrm{og}_{\mathbb{Q}_\ell}^{(1)}$. Then one can put $\mathscr{L}\mathrm{og}_{\mathbb{Q}_\ell}^{(n)} := \mathrm{Sym}^n\mathscr{L}\mathrm{og}_{\mathbb{Q}_\ell}^{(1)}$ with $1^{(n)} := \frac{(1^{(1)})^{\otimes n}}{n!}$.

Definition 8.1.2. The $\mathbb{Q}_\ell$-elliptic logarithm sheaf $\mathscr{L}\mathrm{og}_{\mathbb{Q}_\ell}$ is the pro-sheaf $\mathscr{L}\mathrm{og}_{\mathbb{Q}_\ell} = \varprojlim_n \mathscr{L}\mathrm{og}_{\mathbb{Q}_\ell}^{(n)}$.

We define an action of $\mathscr{H}_{\mathbb{Q}_\ell}$ on $\mathscr{L}\mathrm{og}_{\mathbb{Q}_\ell}$:

$$\mathrm{mult} : \pi^*\mathscr{H}_{\mathbb{Q}_\ell} \otimes \mathscr{L}\mathrm{og}_{\mathbb{Q}_\ell} \to \mathscr{L}\mathrm{og}_{\mathbb{Q}_\ell}. \tag{8.1.4}$$

On $\mathscr{L}\mathrm{og}_{\mathbb{Q}_\ell}^{(n)}$ it is the composition

$$\pi^*\mathscr{H}_{\mathbb{Q}_\ell} \otimes \mathscr{L}\mathrm{og}_{\mathbb{Q}_\ell}^{(n)} \to \pi^*\mathscr{H}_{\mathbb{Q}_\ell} \otimes \mathscr{L}\mathrm{og}_{\mathbb{Q}_\ell}^{(n-1)} \subset \mathscr{L}\mathrm{og}_{\mathbb{Q}_\ell}^{(1)} \otimes \mathscr{L}\mathrm{og}_{\mathbb{Q}_\ell}^{(n-1)} \to \mathscr{L}\mathrm{og}_{\mathbb{Q}_\ell}^{(n)},$$

where the last map is induced by the multiplication

$$\mathscr{L}\mathrm{og}_{\mathbb{Q}_\ell}^{(1)} \otimes \mathrm{Sym}^{n-1}\mathscr{L}\mathrm{og}_{\mathbb{Q}_\ell}^{(1)} \to \mathrm{Sym}^n\mathscr{L}\mathrm{og}_{\mathbb{Q}_\ell}^{(1)}.$$

The most important fact about the logarithm sheaf is the vanishing of its higher direct images except the second one.

Proposition 8.1.3 ([HK99] Lemma A.1.4.)**.** *One has*

$$R^i\pi_*\mathscr{L}\mathrm{og}_{\mathbb{Q}_\ell} = \begin{cases} 0 & \text{if } i \neq 2 \\ R^2\pi_*\mathbb{Q}_\ell \cong \mathbb{Q}_\ell(-1) & \text{if } i = 2 \end{cases}$$

and the isomorphism $R^2\pi_*\mathscr{L}\mathrm{og}_{\mathbb{Q}_\ell} \cong R^2\pi_*\mathbb{Q}_\ell$ is induced by the augmentation $\mathscr{L}\mathrm{og}_{\mathbb{Q}_\ell} \to \mathscr{L}\mathrm{og}_{\mathbb{Q}_\ell}^{(0)} = \mathbb{Q}_\ell$.

Another important fact about the logarithm sheaf is the splitting principle, which we formulate as follows:

Proposition 8.1.4 ([HK99] Corollary A.2.6.)**.** *Let $\varphi : \mathcal{E} \to \mathcal{E}'$ be an isogeny and denote by $\mathcal{L}og'_{\mathbb{Q}_\ell}$ the logarithm sheaf of $\mathcal{E}'$. Then one has an isomorphism*

$$\mathcal{L}og_{\mathbb{Q}_\ell} \cong \varphi^* \mathcal{L}og'_{\mathbb{Q}_\ell}.$$

In particular, for each section $t \in \ker\varphi(S)$ one has a canonical isomorphism

$$t^* \mathcal{L}og_{\mathbb{Q}_\ell} \cong e'^* \mathcal{L}og'_{\mathbb{Q}_\ell} \cong e^* \mathcal{L}og_{\mathbb{Q}_\ell} = \prod_{n \geq 0} \mathrm{Sym}^n \mathscr{H}_{\mathbb{Q}_\ell}.$$

Note that in the case where $\varphi = [N]$ the isomorphism in the proposition induces the multiplication by $[N]^k$ on the graded pieces $\pi^* \mathrm{Sym}^k \mathscr{H}_{\mathbb{Q}_\ell}$ of $\mathcal{L}og_{\mathbb{Q}_\ell}$.

8.2. **The $\mathbb{Z}_\ell$-elliptic logarithm sheaf.** For simplicity we assume that S is connected.

Recall from Example 2.3.2 that $\mathcal{L}_{\mathbb{Z}/\ell^r\mathbb{Z}} = [\ell^r]_* \mathbb{Z}/\ell^r\mathbb{Z}$, where $[\ell^r] : \mathcal{E} \to \mathcal{E}$ is the ℓ^r-multiplication map and that $\mathcal{L}_{\mathbb{Z}_\ell}$ is the projective system

$$\mathcal{L}_{\mathbb{Z}_\ell} = (\mathcal{L}_{\mathbb{Z}/\ell^r\mathbb{Z}})_{r \geq 1}.$$

In this section we compare $\mathcal{L}_{\mathbb{Z}_\ell}$ and $\mathcal{L}og_{\mathbb{Z}_\ell}$.

Fix a geometric point $\overline{s}$ of S and let $\overline{e} := e(\overline{s})$ be the corresponding geometric point of $\mathcal{E}$. One has a cartesian diagram

$$\begin{tikzcd} \mathcal{E}_{\overline{s}} \arrow[r] \arrow[d, shift left, "\pi"] & \mathcal{E} \arrow[d, shift right, "\pi"'] \\ \overline{s} \arrow[r] \arrow[u, bend left, "e"] & S \arrow[u, bend right, "e"'] \end{tikzcd}$$

By Formula (2.3.8) we have

$$\mathcal{L}_{\mathbb{Z}_\ell, \overline{e}} \cong (e^* \mathcal{L}_{\mathbb{Z}_\ell})_{\overline{s}} \cong \Lambda(\mathscr{H}_{\mathbb{Z}_\ell, \overline{s}})$$

and we want to determine explicitly the action of the fundamental group $\pi_1(S, \overline{s})$ on $\Lambda(\mathscr{H}_{\mathbb{Z}_\ell, \overline{s}})$. Let $\pi'_1(\mathcal{E}_{\overline{s}}, \overline{e})$ be the largest pro-ℓ-quotient of $\pi_1(\mathcal{E}_{\overline{s}}, \overline{e})$ and note that

$$\pi'_1(\mathcal{E}_{\overline{s}}, \overline{e}) \cong \mathscr{H}_{\mathbb{Z}_\ell, \overline{s}} = T_\ell \mathcal{E}_{\overline{s}}.$$

Consider $\pi_* : \pi_1(\mathcal{E}, \overline{e}) \to \pi_1(S, \overline{s})$ and let $\ker(\pi_*)/\mathcal{N}$ be the largest pro-ℓ-quotient of $\ker(\pi_*)$. Then we define $\pi'_1(\mathcal{E}, \overline{e}) := \pi_1(\mathcal{E}, \overline{e})/\mathcal{N}$ and one has a split exact sequence of fundamental groups

$$1 \to \pi'_1(\mathcal{E}_{\overline{s}}, \overline{e}) \to \pi'_1(\mathcal{E}, \overline{e}) \xrightarrow{\pi_*} \pi_1(S, \overline{s}) \to 1. \tag{8.2.1}$$

The splitting of the sequence is given by e_*. This induces an isomorphism

$$\pi'_1(\mathcal{E}, \overline{e}) \cong \mathscr{H}_{\mathbb{Z}_\ell, \overline{s}} \rtimes \pi_1(S, \overline{s}).$$

Denote by

$$\varrho : \pi_1(S, \overline{s}) \to \mathrm{Aut}_{\mathbb{Z}_\ell}(\mathscr{H}_{\mathbb{Z}_\ell, \overline{s}})$$

the action of $\pi_1(S,\overline{s})$ on $\mathscr{H}_{\mathbb{Z}_\ell,\overline{s}}$. This coincides with the action of $\pi_1(S,\overline{s})$ given by conjugation with e_*

$$\varrho(g)(h) = e_*(g)he_*(g)^{-1} \tag{8.2.2}$$

for all $g \in \pi_1(S,\overline{s})$ and $h \in \mathscr{H}_{\mathbb{Z}_\ell,\overline{s}}$. By functoriality this induces an action

$$\varrho : \pi_1(S,\overline{s}) \to \mathrm{Aut}_{\mathbb{Z}_\ell}(\Lambda(\mathscr{H}_{\mathbb{Z}_\ell,\overline{s}}))$$

still denoted by ϱ. On the other hand the map $\delta : \mathscr{H}_{\mathbb{Z}_\ell,\overline{s}} \to \Lambda(\mathscr{H}_{\mathbb{Z}_\ell,\overline{s}})^*$, defined by $h \mapsto \delta_h$ is an action of $\mathscr{H}_{\mathbb{Z}_\ell,\overline{s}}$ on $\Lambda(\mathscr{H}_{\mathbb{Z}_\ell,\overline{s}})$ by multiplication with δ_h, i.e., by "translation" with h.

Proposition 8.2.1. *The action of $\pi_1'(\mathcal{E},\overline{e})$ on*

$$\mathcal{L}_{\mathbb{Z}_\ell,\overline{e}} \cong \Lambda(\mathscr{H}_{\mathbb{Z}_\ell,\overline{s}})$$

is given by the group homomorphism

$$\begin{aligned} \mathscr{H}_{\mathbb{Z}_\ell,\overline{s}} \rtimes \pi_1(S,\overline{s}) &\to \mathrm{Aut}_{\mathbb{Z}_\ell}(\Lambda(\mathscr{H}_{\mathbb{Z}_\ell,\overline{s}})) \\ (h,g) &\mapsto \delta_h\varrho(g). \end{aligned}$$

Proof. The map is a group homomorphism by the relation in Formula (8.2.2). The étale covering $[\ell^r] : \mathcal{E} \to \mathcal{E}$ belongs to the subgroup

$$\ell^r\mathscr{H}_{\mathbb{Z}_\ell,\overline{s}} \rtimes \pi_1(S,\overline{s})$$

of $\mathscr{H}_{\mathbb{Z}_\ell,\overline{s}} \rtimes \pi_1(S,\overline{s})$. By general principles the lisse sheaf $\mathcal{L}_{\mathbb{Z}/\ell^r\mathbb{Z}} = [\ell^r]_*\mathbb{Z}/\ell^r\mathbb{Z}$ corresponds to the representation

$$\mathscr{H}_{\mathbb{Z}_\ell,\overline{s}} \rtimes \pi_1(S,\overline{s}) \to \mathrm{Aut}_{\mathbb{Z}/\ell^r\mathbb{Z}}(\mathbb{Z}/\ell^r\mathbb{Z}[\mathscr{H}_{\mathbb{Z}/\ell^r\mathbb{Z},\overline{s}}])$$

on $\mathcal{L}\mathrm{og}_{\mathbb{Z}/\ell^r\mathbb{Z},\overline{e}} \cong \mathbb{Z}/\ell^r\mathbb{Z}[\mathscr{H}_{\mathbb{Z}/\ell^r\mathbb{Z},\overline{s}}]$ which is induced from the trivial representation of $\ell^r\mathscr{H}_{\mathbb{Z}_\ell,\overline{s}} \rtimes \pi_1(S,\overline{s})$ on $\mathbb{Z}/\ell^r\mathbb{Z}$. This induced action is exactly the one described in the proposition. Taking the projective limit over r gives the desired result. □

Corollary 8.2.2. *Let $I := I(\mathscr{H}_{\mathbb{Z}_\ell,\overline{s}}) := \ker(\Lambda(\mathscr{H}_{\mathbb{Z}_\ell,\overline{s}}) \xrightarrow{\int} \mathbb{Z}_\ell)$ be the augmentation ideal. Then I^k is stable under the action of $\mathscr{H}_{\mathbb{Z}_\ell,\overline{s}} \rtimes \pi_1(S,\overline{s})$ and on*

$$I^k/I^{k+1} \cong \mathrm{Sym}^n\mathscr{H}_{\mathbb{Z}_\ell,\overline{s}}$$

the subgroup $\mathscr{H}_{\mathbb{Z}_\ell,\overline{s}}$ acts trivially.

Proof. That I^k is stable under $\mathscr{H}_{\mathbb{Z}_\ell,\overline{s}}$ and that $\mathscr{H}_{\mathbb{Z}_\ell,\overline{s}}$ acts trivially on I^k/I^{k+1} is Lemma 1.3.1. Obviously, $\pi_1(S,\overline{s})$ acts via ring isomorphisms and so also preserves the augmentation ideal. □

Definition 8.2.3. We denote by $A^n\mathcal{L}_{\mathbb{Z}_\ell}$ the subsheaf of $\mathcal{L}_{\mathbb{Z}_\ell}$ corresponding to the $\mathscr{H}_{\mathbb{Z}_\ell,\overline{s}} \rtimes \pi_1(S,\overline{s})$-representation $I^n \subset \Lambda(\mathscr{H}_{\mathbb{Z}_\ell,\overline{s}})$ and let

$$\mathcal{L}^{(n)}_{\mathbb{Z}_\ell} := \mathcal{L}_{\mathbb{Z}_\ell}/A^{n+1}\mathcal{L}_{\mathbb{Z}_\ell}$$

be the quotient sheaf.

Note that $\mathcal{L}^{(n)}_{\mathbb{Z}_\ell} = (\mathcal{L}^{(n)}_{\mathbb{Z}/\ell^r\mathbb{Z}})_r$, where $\mathcal{L}^{(n)}_{\mathbb{Z}/\ell^r\mathbb{Z}}$ is the quotient of $\mathcal{L}_{\mathbb{Z}/\ell^r\mathbb{Z}}$ by the image of $A^{n+1}\mathcal{L}_{\mathbb{Z}_\ell}$. The images of the sheaves $A^k\mathcal{L}_{\mathbb{Z}_\ell}$ define an unipotent filtration of length n on $\mathcal{L}^{(n)}_{\mathbb{Z}_\ell}$ and one has an exact sequence

$$0 \to \pi^*\mathrm{Sym}^n \mathscr{H}_{\mathbb{Z}_\ell} \to \mathcal{L}^{(n+1)}_{\mathbb{Z}_\ell} \to \mathcal{L}^{(n)}_{\mathbb{Z}_\ell} \to 0.$$

The element $1 = \delta_0 \in \Lambda(\mathscr{H}_{\mathbb{Z}_\ell,\overline{s}})$ defines

$$1^{(n)} \in \mathrm{Hom}_S(\mathbb{Z}_\ell, e^*\mathcal{L}^{(n)}_{\mathbb{Z}_\ell}).$$

Theorem 8.2.4. *The pair $(\mathcal{L}^{(n)}_{\mathbb{Z}_\ell}, 1^{(n)})$ is canonically isomorphic to $(\mathcal{L}og^{(n)}_{\mathbb{Z}_\ell}, 1^{(n)})$. In particular, the $\mathbb{Q}_\ell$-sheaf associated to $(\mathcal{L}^{(n)}_{\mathbb{Z}_\ell}, 1^{(n)})$ is $(\mathcal{L}og^{(n)}_{\mathbb{Q}_\ell}, 1^{(n)})$.*

Proof. We check that the map $\nu^\cdot$ from Formula (8.1.2) is an isomorphism. By definition

$$\mathrm{gr}^k_A(\mathcal{L}_{\mathbb{Z}_\ell}/A^{n+1}\mathcal{L}_{\mathbb{Z}_\ell}) \cong \pi^*\mathrm{Sym}^k\mathscr{H}_{\mathbb{Z}_\ell}$$

so that $\pi_*\mathrm{gr}^k_A(\mathcal{L}_{\mathbb{Z}_\ell}/A^{n+1}\mathcal{L}_{\mathbb{Z}_\ell}) \cong \mathrm{Sym}^k\mathscr{H}_{\mathbb{Z}_\ell}$. By definition of $1^{(n)}$ the isomorphism

$$\mathrm{Sym}^\cdot\mathscr{H}_F/\mathrm{Sym}^{\geq n+1}\mathscr{H}_F \cong \pi_*\mathrm{gr}^\cdot_A(\mathcal{L}_{\mathbb{Z}_\ell}/A^{n+1}\mathcal{L}_{\mathbb{Z}_\ell})$$

maps 1 to $1^{(n)}$. $\square$

The theorem allows to compare cohomology classes with values in $\mathcal{L}_{\mathbb{Z}_\ell}$ and $\mathcal{L}og_{\mathbb{Q}_\ell}$. We will be later in the following situation: fix an integer $c > 1$ and consider the restriction of $\mathcal{L}_{\mathbb{Z}_\ell}$ and $\mathcal{L}og_{\mathbb{Q}_\ell}$ to $\mathcal{E} \setminus \mathcal{E}[c]$.

Corollary 8.2.5. *There is a comparison morphism*

$$\mathrm{comp} : H^1(\mathcal{E} \setminus \mathcal{E}[c], \mathcal{L}_{\mathbb{Z}_\ell}(1)) \to H^1(\mathcal{E} \setminus \mathcal{E}[c], \mathcal{L}og_{\mathbb{Q}_\ell}(1)).$$

Proof. For all n the morphism $\mathcal{L}_{\mathbb{Z}_\ell} \to \mathcal{L}^{(N)}_{\mathbb{Z}_\ell} \cong \mathcal{L}og^{(n)}_{\mathbb{Z}_\ell}$ induces

$$H^1(\mathcal{E} \setminus \mathcal{E}[c], \mathcal{L}og_{\mathbb{Z}_\ell}(1)) \to H^1(\mathcal{E} \setminus \mathcal{E}[c], \mathcal{L}og^{(n)}_{\mathbb{Z}_\ell}(1)) \to H^1(\mathcal{E} \setminus \mathcal{E}[c], \mathcal{L}og^{(n)}_{\mathbb{Q}_\ell}(1))$$

where the last map is the one from (2.1.1). Taking the limit over n and observing formula (2.1.2) gives the desired map. $\square$

Finally, we relate the splitting principle and the moment map. Recall from Formula (2.3.7) that

$$t^*\mathcal{L}_{\mathbb{Z}_\ell} \cong \Lambda(\mathscr{H}_{\mathbb{Z}_\ell}\langle t\rangle).$$

If we pull-back both sides of the comparison map with an N-torsion point $t : S \to \mathcal{E} \setminus \mathcal{E}[c]$ we get

$$\mathrm{comp} : H^1(S, \Lambda(\mathscr{H}_{\mathbb{Z}_\ell}\langle t\rangle)(1)) \to H^1(S, \mathcal{L}og_{\mathbb{Q}_\ell}(1)).$$

Proposition 8.2.6. *Let t be an N-torsion section $t : S \to \mathcal{E} \setminus \mathcal{E}[c]$. There is a commutative diagram*

$$\begin{tikzcd}
H^1(S, \Lambda(\mathscr{H}_{\mathbb{Z}_\ell}\langle t\rangle)(1)) \arrow[r, "\mathrm{comp}"] \arrow[dd, "\widetilde{\mathrm{mom}}_t^k"'] \arrow[ddr, "\mathrm{mom}_t^k"'] & H^1(S, t^*\mathcal{L}\mathrm{og}_{\mathbb{Q}_\ell}(1)) \arrow[d, "\cong"] \\
 & H^1(S, \prod_{k\geq 0} \mathrm{Sym}^k \mathscr{H}_{\mathbb{Q}_\ell}(1)) \arrow[d, "\mathrm{pr}_k"] \\
H^1(S, \mathrm{Sym}^k \mathscr{H}_{\mathbb{Z}_\ell}(1)) \arrow[r, "\frac{1}{k!N^k}"] & H^1(S, \mathrm{Sym}^k \mathscr{H}_{\mathbb{Q}_\ell}(1)),
\end{tikzcd}$$

where the upper right vertical arrow is induced from Proposition 8.1.4 and the lower horizontal arrow is multiplication with $\frac{1}{k!N^k}$.

Proof. For the proof we have to consider the analogue of the isomorphism in Proposition 8.1.4 in the $\mathbb{Z}_\ell$ situation. The trace map induces a morphism of sheaves $[N]_*\mathcal{L}_{\mathbb{Z}_\ell} \to \mathcal{L}_{\mathbb{Z}_\ell}$, hence a morphism

$$\mathcal{L}_{\mathbb{Z}_\ell} \to [N]^*\mathcal{L}_{\mathbb{Z}_\ell}.$$

If we pull-back with t we get a morphism $[N]_! : \Lambda(\mathscr{H}_{\mathbb{Z}_\ell}\langle t\rangle) \to \Lambda(\mathscr{H}_{\mathbb{Z}_\ell})$ and a commutative diagram

$$\begin{tikzcd}
H^1(S, \Lambda(\mathscr{H}_{\mathbb{Z}_\ell}\langle t\rangle)(1)) \arrow[r, "\mathrm{comp}"] \arrow[d, "{[N]_!}"'] & H^1(S, t^*\mathcal{L}\mathrm{og}_{\mathbb{Q}_\ell}(1)) \arrow[d, "\cong"] \\
H^1(S, \Lambda(\mathscr{H}_{\mathbb{Z}_\ell})(1)) \arrow[r, "\mathrm{comp}"] & H^1(S, \prod_{n\geq 0} \mathrm{Sym}^n \mathscr{H}_{\mathbb{Q}_\ell}(1)).
\end{tikzcd}$$

The map comp corresponds on the stalk at $\overline{s}$ to the ring homomorphism

$$\mathrm{mom} : \Lambda(\mathscr{H}_{\mathbb{Z}_\ell,\overline{s}}) \to \mathfrak{U}(\mathscr{H}_{\mathbb{Q}_\ell,\overline{s}})$$

from Proposition 1.3.2 so that the composition with pr_k is mom^k. $\square$

8.3. **The elliptic polylogarithm.** The Leray spectral sequence together with Proposition 8.1.3 and the localization sequence gives an isomorphism

$$\mathrm{Ext}^1_{\mathcal{E}\setminus\{e\}}(\pi^*\mathscr{H}_{\mathbb{Q}_\ell}, \mathcal{L}\mathrm{og}_{\mathbb{Q}_\ell}(1)) \cong \mathrm{Hom}_S(\mathscr{H}_{\mathbb{Q}_\ell}, \prod_{n\geq 1} \mathrm{Sym}^n \mathscr{H}_{\mathbb{Q}_\ell}) \tag{8.3.1}$$

(see [HK99, A.3]).

Definition 8.3.1. The (small) elliptic polylogarithm is the class

$$\mathrm{pol} \in \mathrm{Ext}^1_{\mathcal{E}\setminus\{e\}}(\pi^*\mathscr{H}_{\mathbb{Q}_\ell}, \mathcal{L}\mathrm{og}_{\mathbb{Q}_\ell}(1)),$$

which maps to the canonical inclusion $\mathscr{H}_{\mathbb{Q}_\ell} \hookrightarrow \prod_{n\geq 1} \mathrm{Sym}^n \mathscr{H}_{\mathbb{Q}_\ell}$ under the above isomorphism (8.3.1).

For the explicit construction of the elliptic polylogarithm a slight variant of the isomorphism (8.3.1) is useful. The localization sequence gives for $c > 1$

$$0 \to H^1(\mathcal{E} \setminus \mathcal{E}[c], \mathcal{L}og_{\mathbb{Q}_\ell}(1)) \to H^0(\mathcal{E}[c], \mathcal{L}og_{\mathbb{Q}_\ell} |_{\mathcal{E}[c]}) \to H^2(\mathcal{E}, \mathcal{L}og_{\mathbb{Q}_\ell}(1)) \to 0$$

and $H^2(\mathcal{E}, \mathcal{L}og_{\mathbb{Q}_\ell}(1)) \cong H^2(\mathcal{E}, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell$ by Proposition 8.1.3. We have

$$H^0(\mathcal{E}[c], \mathbb{Q}_\ell) \subset H^0(\mathcal{E}[c], \mathcal{L}og_{\mathbb{Q}_\ell} |_{\mathcal{E}[c]})$$

and as $\mathcal{E}[c] = (\mathcal{E}[c] \setminus \{e\}) \amalg \{e\}$ we have two sections in $H^0(\mathcal{E}[c], \mathbb{Q}_\ell)$: the first is $1_{\mathcal{E}[c]}$, which is identically 1 on all of $\mathcal{E}[c]$ and the second $1_{\{e\}}$, which is identically 1 on $\{e\}$ and zero on $\mathcal{E}[c] \setminus \{e\}$. Note that

$$c^2 1_{\{e\}} - 1_{\mathcal{E}[c]} \in H^0(\mathcal{E}[c], \mathbb{Q}_\ell)$$

maps to 0 in $H^2(\mathcal{E}, \mathcal{L}og_{\mathbb{Q}_\ell}(1)) \cong \mathbb{Q}_\ell$.

Definition 8.3.2. The polylogarithm associated to $c^2 1_{\{e\}} - 1_{\mathcal{E}[c]}$ is the cohomology class

$$\mathrm{pol}_{c^2 1_{\{e\}} - 1_{\mathcal{E}[c]}} \in H^1(\mathcal{E} \setminus \mathcal{E}[c], \mathcal{L}og_{\mathbb{Q}_\ell}(1)) \tag{8.3.2}$$

mapping to $c^2 1_{\{e\}} - 1_{\mathcal{E}[c]} \in H^0(\mathcal{E}[c], \mathbb{Q}_\ell)$.

This cohomology class is related to pol as follows. Write

$$H^1(\mathcal{E} \setminus \mathcal{E}[c], \mathcal{L}og_{\mathbb{Q}_\ell}(1)) \cong \mathrm{Ext}^1_{\mathcal{E} \setminus \mathcal{E}[c]}(\mathbb{Q}_\ell, \mathcal{L}og_{\mathbb{Q}_\ell}(1))$$

and define a map

$$\mathrm{mult}_{\mathscr{H}_{\mathbb{Q}_\ell}} : \mathrm{Ext}^1_{\mathcal{E} \setminus \mathcal{E}[c]}(\mathbb{Q}_\ell, \mathcal{L}og_{\mathbb{Q}_\ell}(1)) \to \mathrm{Ext}^1_{\mathcal{E} \setminus \mathcal{E}[c]}(\pi^* \mathscr{H}_{\mathbb{Q}_\ell}, \mathcal{L}og_{\mathbb{Q}_\ell}(1)) \tag{8.3.3}$$

by first tensoring an extension with $\pi^* \mathscr{H}_{\mathbb{Q}_\ell}$ and then push-out with mult : $\pi^* \mathscr{H}_{\mathbb{Q}_\ell} \otimes \mathcal{L}og_{\mathbb{Q}_\ell} \to \mathcal{L}og_{\mathbb{Q}_\ell}$ from Equation (8.1.4). This gives

$$\mathrm{mult}_{\mathscr{H}_{\mathbb{Q}_\ell}}(\mathrm{pol}_{c^2 1_{\{e\}} - 1_{\mathcal{E}[c]}}) \in \mathrm{Ext}^1_{\mathcal{E} \setminus \mathcal{E}[c]}(\pi^* \mathscr{H}_{\mathbb{Q}_\ell}, \mathcal{L}og_{\mathbb{Q}_\ell}(1)).$$

On the other hand consider

$$[c]^* \mathrm{pol} \in \mathrm{Ext}^1_{\mathcal{E} \setminus \mathcal{E}[c]}(\pi^* \mathscr{H}_{\mathbb{Q}_\ell}, [c]^* \mathcal{L}og_{\mathbb{Q}_\ell}(1))$$

and use the isomorphism $\mathcal{L}og_{\mathbb{Q}_\ell} \cong [c]^* \mathcal{L}og_{\mathbb{Q}_\ell}$ from Proposition 8.1.4 to obtain a class in $\mathrm{Ext}^1_{\mathcal{E} \setminus \mathcal{E}[c]}(\pi^* \mathscr{H}_{\mathbb{Q}_\ell}, \mathcal{L}og_{\mathbb{Q}_\ell}(1))$. Restriction of pol to $\mathcal{E} \setminus \mathcal{E}[c]$ gives another class in $\mathrm{Ext}^1_{\mathcal{E} \setminus \mathcal{E}[c]}(\pi^* \mathscr{H}_{\mathbb{Q}_\ell}, \mathcal{L}og_{\mathbb{Q}_\ell}(1))$.

Proposition 8.3.3. *There is an equality*

$$\mathrm{mult}_{\mathscr{H}_{\mathbb{Q}_\ell}}(\mathrm{pol}_{c^2 1_{\{e\}} - 1_{\mathcal{E}[c]}}) = c^2 \mathrm{pol} |_{\mathcal{E} \setminus \mathcal{E}[c]} - c[c]^* \mathrm{pol}$$

in $\mathrm{Ext}^1_{\mathcal{E} \setminus \mathcal{E}[c]}(\pi^* \mathscr{H}_{\mathbb{Q}_\ell}, \mathcal{L}og_{\mathbb{Q}_\ell}(1))$.

Proof. As in Equation (8.3.1) we have

$$\mathrm{Ext}^1_{\mathcal{E}\setminus\mathcal{E}[c]}(\pi^*\mathscr{H}_{\mathbb{Q}_\ell}, \mathcal{L}\mathrm{og}_{\mathbb{Q}_\ell}(1)) \subset \mathrm{Hom}_{\mathcal{E}[c]}(\mathscr{H}_{\mathbb{Q}_\ell}, \prod_{n\geq 0}\mathrm{Sym}^n\mathscr{H}_{\mathbb{Q}_\ell})$$

and we have to show that the images of the elements $\mathrm{mult}_{\mathscr{H}_{\mathbb{Q}_\ell}}(\mathrm{pol}_{c^2 1_{\{e\}}-1_{\mathcal{E}[c]}})$ and $c^2\mathrm{pol}\,|_{\mathcal{E}\setminus\mathcal{E}[c]} - c[c]^*\mathrm{pol}$ in the right hand side are the same. The right hand side contains

$$\mathrm{Hom}_{\mathcal{E}[c]}(\mathscr{H}_{\mathbb{Q}_\ell}, \mathscr{H}_{\mathbb{Q}_\ell})$$

and by definition the images of both elements are already contained in this group. In $\mathrm{Hom}_{\mathcal{E}[c]}(\mathscr{H}_{\mathbb{Q}_\ell}, \mathscr{H}_{\mathbb{Q}_\ell})$ we have two elements: the first is $\mathrm{id}_{\mathcal{E}[c]}$, which has constant value the identity map on $\mathcal{E}[c]$ and the second is $\mathrm{id}_{\{e\}}$, which maps $\{e\}$ to the identity map and $\mathcal{E}[c]\setminus\{e\}$ to zero. It follows directly from the definition of $\mathrm{mult}_{\mathscr{H}_{\mathbb{Q}_\ell}}(\mathrm{pol}_{c^2 1_{\{e\}}-1_{\mathcal{E}[c]}})$ and $c^2\mathrm{pol}\,|_{\mathcal{E}\setminus\mathcal{E}[c]} - c[c]^*\mathrm{pol}$ that both elements map to

$$c^2\mathrm{id}_{\{e\}} - \mathrm{id}_{\mathcal{E}[c]} \in \mathrm{Hom}_{\mathcal{E}[c]}(\mathscr{H}_{\mathbb{Q}_\ell}, \mathscr{H}_{\mathbb{Q}_\ell})$$

(note that the identification $\mathcal{L}\mathrm{og}_{\mathbb{Q}_\ell} \cong [c]^*\mathcal{L}\mathrm{og}_{\mathbb{Q}_\ell}$ is multiplication by c on $\mathscr{H}_{\mathbb{Q}_\ell}$ so that the residue of $[c]^*\mathrm{pol}$ is $\frac{1}{c}\mathrm{id}_{\mathcal{E}[c]}$). □

8.4. **Eisenstein classes.** In this section we recall the definition of the Eisenstein classes. Consider

$$\mathrm{pol} \in \mathrm{Ext}^1_{\mathcal{E}\setminus\{e\}}(\pi^*\mathscr{H}_{\mathbb{Q}_\ell}, \mathcal{L}\mathrm{og}_{\mathbb{Q}_\ell}(1))$$

and a non-zero N-torsion section $t \in \mathcal{E}(S)$. If we use the isomorphism $t^*\mathcal{L}\mathrm{og}_{\mathbb{Q}_\ell} \cong \prod_{n\geq 0}\mathrm{Sym}^n\mathscr{H}_{\mathbb{Q}_\ell}$ from Proposition 8.1.4 we get

$$t^*\mathrm{pol} = (t^*\mathrm{pol}^n)_{n\geq 0} \in \mathrm{Ext}^1_S(\mathscr{H}_{\mathbb{Q}_\ell}, \prod_{n\geq 0}\mathrm{Sym}^n\mathscr{H}_{\mathbb{Q}_\ell}(1)).$$

To get classes in $H^1(S, \mathrm{Sym}^n\mathscr{H}_{\mathbb{Q}_\ell}(1))$ we use the map

(8.4.1)

$$\mathrm{contr}_{\mathscr{H}_{\mathbb{Q}_\ell}} : \mathrm{Ext}^1_S(\mathscr{H}_{\mathbb{Q}_\ell}, \prod_{n\geq 0}\mathrm{Sym}^n\mathscr{H}_{\mathbb{Q}_\ell}(1)) \to \mathrm{Ext}^1_S(\mathbb{Q}_\ell, \prod_{n\geq 1}\mathrm{Sym}^{n-1}\mathscr{H}_{\mathbb{Q}_\ell}(1))$$

defined by first tensoring an extension with $\mathscr{H}^\vee_{\mathbb{Q}_\ell}$, where $\mathscr{H}^\vee_{\mathbb{Q}_\ell}$ is the dual of $\mathscr{H}_{\mathbb{Q}_\ell}$, and then compose with the contraction map

$$\mathscr{H}^\vee_{\mathbb{Q}_\ell} \otimes \mathrm{Sym}^n\mathscr{H}_{\mathbb{Q}_\ell} \to \mathrm{Sym}^{n-1}\mathscr{H}_{\mathbb{Q}_\ell}$$

mapping $h^\vee \otimes h_1 \otimes \cdots \otimes h_n$ to $\frac{1}{n+1}\sum_{j=1}^n h^\vee(h_j)h_1 \otimes \cdots \widehat{h_j} \cdots \otimes h_n$.

Definition 8.4.1. Let $N > 1$ and $t \in \mathcal{E}[N](S)$ be a non-zero N-torsion point, then

$$\mathrm{Eis}^k_{\mathbb{Q}_\ell}(t) := -N^{k-1}\mathrm{contr}_{\mathscr{H}_{\mathbb{Q}_\ell}}(t^*\mathrm{pol}^{k+1}) \in H^1(S, \mathrm{Sym}^k\mathscr{H}_{\mathbb{Q}_\ell}(1))$$

is called the k-th Eisenstein class. If $\psi : (\mathcal{E}[N](S) \setminus \{e\}) \to \mathbb{Q}$ is a map, we define

$$\mathrm{Eis}^k_{\mathbb{Q}_\ell}(\psi) := \sum_{t \in \mathcal{E}[N](S)\setminus\{e\}} \psi(t)\mathrm{Eis}^k_{\mathbb{Q}_\ell}(t).$$

Remark 8.4.2. The factor $-N^{k-1}$ is for historical reasons as the Eisenstein classes were originally defined in a different way by Beilinson (see [Beĭ86, Theorem 7.3])

Let $(c, N) = 1$ and recall from Definition 8.3.2 the class

$$\mathrm{pol}_{c^2 1_{\{e\}} - 1_{\mathcal{E}[c]}} \in \mathrm{Ext}^1_{\mathcal{E}\setminus\mathcal{E}[c]}(\mathbb{Q}_\ell, \mathcal{L}\mathrm{og}_{\mathbb{Q}_\ell}(1)).$$

If we pull this back along a non-zero N-torsion section $t \in \mathcal{E}[N](S)$, we get, using again $t^*\mathcal{L}\mathrm{og}_{\mathbb{Q}_\ell} \cong \prod_{n\geq 0} \mathrm{Sym}^n \mathscr{H}_{\mathbb{Q}_\ell}$,

$$t^*\mathrm{pol}_{c^2 1_{\{e\}} - 1_{\mathcal{E}[c]}} \in \mathrm{Ext}^1_S(\mathbb{Q}_\ell, \prod_{n\geq 0} \mathrm{Sym}^n \mathscr{H}_{\mathbb{Q}_\ell}(1))$$

and the k-th component gives a class

$$t^*\mathrm{pol}^k_{c^2 1_{\{e\}} - 1_{\mathcal{E}[c]}} \in H^1(S, \mathrm{Sym}^k \mathscr{H}_{\mathbb{Q}_\ell}(1)).$$

Lemma 8.4.3. *In $H^1(S, \mathrm{Sym}^k \mathscr{H}_{\mathbb{Q}_\ell}(1))$ we have the equality*

$$t^*\mathrm{pol}^k_{c^2 1_{\{e\}} - 1_{\mathcal{E}[c]}} = \frac{-1}{N^{k-1}}(c^2 \mathrm{Eis}^k_{\mathbb{Q}_\ell}(t) - c^{-k}\mathrm{Eis}^k_{\mathbb{Q}_\ell}([c]t)).$$

In particular, for $c \equiv 1 \mod N$ one has

$$t^*\mathrm{pol}^k_{c^2 1_{\{e\}} - 1_{\mathcal{E}[c]}} = \frac{-1}{N^{k-1}} \frac{c^{k+2}-1}{c^k} \mathrm{Eis}^k_{\mathbb{Q}_\ell}(t).$$

Proof. According to Proposition 8.3.3 we have

$$\mathrm{mult}_{\mathscr{H}_{\mathbb{Q}_\ell}}(\mathrm{pol}^k_{c^2 1_{\{e\}} - 1_{\mathcal{E}[c]}}) = c^2 \mathrm{pol}^{k+1} |_{\mathcal{E}\setminus\mathcal{E}[c]} - c[c]^*\mathrm{pol}^{k+1}.$$

Taking the pull-back along t of the right hand side and applying the map $\mathrm{contr}_{\mathscr{H}_{\mathbb{Q}_\ell}}$ gives

$$\frac{-1}{N^{k-1}}(c^2 \mathrm{Eis}^k_{\mathbb{Q}_\ell}(t) - c^{-k}\mathrm{Eis}^k_{\mathbb{Q}_\ell}([c]t))$$

(note that the isomorphism $\mathcal{L}\mathrm{og}_{\mathbb{Q}_\ell} \cong [c]^*\mathcal{L}\mathrm{og}_{\mathbb{Q}_\ell}$ is multiplication by c^{k+1} on $\mathrm{Sym}^{k+1}\mathscr{H}_{\mathbb{Q}_\ell}$ by the remark after Proposition 8.1.4 so that we have to divide by c^{k+1}). Thus it remains to show that

$$\mathrm{contr}_{\mathscr{H}_{\mathbb{Q}_\ell}}(t^*\mathrm{mult}_{\mathscr{H}_{\mathbb{Q}_\ell}}(\mathrm{pol}^k_{c^2 1_{\{e\}} - 1_{\mathcal{E}[c]}})) = t^*\mathrm{pol}^k_{c^2 1_{\{e\}} - 1_{\mathcal{E}[c]}}.$$

But obviously we have $\mathrm{contr}_{\mathscr{H}_{\mathbb{Q}_\ell}} \circ t^*\mathrm{mult}_{\mathscr{H}_{\mathbb{Q}_\ell}} = \mathrm{contr}_{\mathscr{H}_{\mathbb{Q}_\ell}} \circ \mathrm{mult}_{\mathscr{H}_{\mathbb{Q}_\ell}} t^*$, where the last $\mathrm{mult}_{\mathscr{H}_{\mathbb{Q}_\ell}}$ is now on $\prod_{n\geq 0} \mathrm{Sym}^n \mathscr{H}_{\mathbb{Q}_\ell}$, which gives

$$\mathrm{contr}_{\mathscr{H}_{\mathbb{Q}_\ell}}(t^*\mathrm{mult}_{\mathscr{H}_{\mathbb{Q}_\ell}}(\mathrm{pol}^k_{c^2 1_{\{e\}} - 1_{\mathcal{E}[c]}})) = \mathrm{contr}_{\mathscr{H}_{\mathbb{Q}_\ell}} \circ \mathrm{mult}_{\mathscr{H}_{\mathbb{Q}_\ell}}(t^*\mathrm{pol}^k_{c^2 1_{\{e\}} - 1_{\mathcal{E}[c]}}).$$

A direct check shows that $\mathrm{contr}_{\mathscr{H}_{\mathbb{Q}_\ell}} \circ \mathrm{mult}_{\mathscr{H}_{\mathbb{Q}_\ell}}$ is the identity map. This gives the desired result. $\square$

8.5. **Construction of the integral ℓ-adic polylogarithm.** We work with the universal elliptic curve $\pi : \mathcal{E} \to Y(N)$ (in fact one can use any other base, but this is the universal case). Let $c \in \mathbb{Z}$ with $(c, 6\ell N) = 1$ and consider the finite étale map

$$[\ell^r] : \mathcal{E} \setminus \mathcal{E}[\ell^r c] \to \mathcal{E} \setminus \mathcal{E}[c].$$

On $\mathcal{E} \setminus \mathcal{E}[\ell^r c]$ we have the elliptic unit ${}_c\vartheta_{\mathcal{E}}$ from Theorem 5.1.1 and we can consider its image under the Kummer map

$$\partial_r : \mathbb{G}_m(\mathcal{E} \setminus \mathcal{E}[\ell^r c]) \to H^1(\mathcal{E} \setminus \mathcal{E}[\ell^r c], \mathbb{Z}/\ell^r\mathbb{Z}(1)),$$

which we denote by

$$\Theta_{c,r} := \partial_r({}_c\vartheta_{\mathcal{E}}) \in H^1(\mathcal{E} \setminus \mathcal{E}[\ell^r c], \mathbb{Z}/\ell^r\mathbb{Z}(1)). \tag{8.5.1}$$

We identify as in Lemma 3.2.2

$$H^1(\mathcal{E} \setminus \mathcal{E}[\ell^r c], \mathbb{Z}/\ell^r\mathbb{Z}(1)) \cong H^1(\mathcal{E} \setminus \mathcal{E}[c], \mathcal{L}_{\mathbb{Z}/\ell^r\mathbb{Z}}(1)).$$

As $[\ell]_*({}_c\vartheta_{\mathcal{E}}) = {}_c\vartheta_{\mathcal{E}}$ we get $[\ell]_*(\Theta_{c,r}) = \Theta_{c,r-1}$. Taking the inverse limit, we can define a class

$$\begin{aligned}\Theta_c := \varprojlim_r \Theta_{c,r} &\in \varprojlim_r H^1(\mathcal{E} \setminus \mathcal{E}[c], \mathcal{L}_{\mathbb{Z}/\ell^r\mathbb{Z}}(1)) \\ &= H^1(\mathcal{E} \setminus \mathcal{E}[c], \mathcal{L}_{\mathbb{Z}_\ell}(1)).\end{aligned} \tag{8.5.2}$$

Recall from Formula (5.3.2) the class

$$\mathcal{E}\mathcal{S}_c^{\langle t\rangle} := \varprojlim_r \partial_r({}_c\vartheta_{\mathcal{E}}) \in H^1(S, \Lambda(\mathscr{H}_{\mathbb{Z}_\ell}\langle t\rangle)(1)).$$

Lemma 8.5.1. *Let $S = Y(N)$, $(c, N) = 1$ and $t \in \mathcal{E}[N](S) \setminus \{e\}$ be an N-torsion section. Then*

$$t^*\Theta_c = \mathcal{E}\mathcal{S}_c^{\langle t\rangle} \in H^1(Y(N), \Lambda(\mathscr{H}_{\mathbb{Z}_\ell}\langle t\rangle)(1)).$$

Proof. We have $t^*\Theta_c = \partial_r({}_c\vartheta_{\mathcal{E}} \mid_{\mathcal{E}[\ell^r]\langle t\rangle})$ so that the formula is clear from the definitions. □

Recall from Corollary 8.2.5 the homomorphism

$$\mathrm{comp} : H^1(\mathcal{E} \setminus \mathcal{E}[c], \mathcal{L}_{\mathbb{Z}_\ell}(1)) \to H^1(\mathcal{E} \setminus \mathcal{E}[c], \mathcal{L}og_{\mathbb{Q}_\ell}(1)). \tag{8.5.3}$$

We have $\mathrm{comp}(\Theta_c) \in H^1(\mathcal{E} \setminus \mathcal{E}[c], \mathcal{L}og_{\mathbb{Q}_\ell}(1))$. On the other hand, recall from Definition 8.3.2 the class

$$\mathrm{pol}_{c^2 1_{\{e\}} - 1_{\mathcal{E}[c]}} \in H^1(\mathcal{E} \setminus \mathcal{E}[c], \mathcal{L}og_{\mathbb{Q}_\ell}(1)).$$

Theorem 8.5.2. *Let $(c, 6\ell N) = 1$, then*

$$\mathrm{comp}(\Theta_c) = \mathrm{pol}_{c^2 1_{\{e\}} - 1_{\mathcal{E}[c]}} \in H^1(\mathcal{E} \setminus \mathcal{E}[c], \mathcal{L}og_{\mathbb{Q}_\ell}(1)).$$

Proof. Consider the commutative diagram

$$\begin{tikzcd} H^1(\mathcal{E}\setminus\mathcal{E}[c],\mathcal{L}_{\mathbb{Z}_\ell}(1)) \arrow[r, "\mathrm{comp}"] \arrow[d] & H^1(\mathcal{E}\setminus\mathcal{E}[c],\mathcal{L}og_{\mathbb{Q}_\ell}(1)) \arrow[d, hook] \\ H^0(\mathcal{E}[c],\mathcal{L}_{\mathbb{Z}_\ell}\mid_{\mathcal{E}[c]}) \arrow[r, "\mathrm{comp}"] & H^0(\mathcal{E}[c],\mathcal{L}og_{\mathbb{Q}_\ell}\mid_{\mathcal{E}[c]}). \end{tikzcd}$$

By definition of $\mathrm{pol}_{c^2 1_{\{e\}}-1_{\mathcal{E}[c]}}$ its image in $H^0(\mathcal{E}[c],\mathcal{L}og_{\mathbb{Q}_\ell}\mid_{\mathcal{E}[c]})$ is the element

$$c^2 1_{\{e\}}-1_{\mathcal{E}[c]}\in H^0(\mathcal{E}[c],\mathbb{Q}_\ell)\subset H^0(\mathcal{E}[c],\mathcal{L}og_{\mathbb{Q}_\ell}\mid_{\mathcal{E}[c]}).$$

To conclude the proof of Theorem 8.5.2 it suffices to compute the image of Θ_c in $H^0(\mathcal{E}[c],\mathcal{L}og_{\mathbb{Q}_\ell}\mid_{\mathcal{E}[c]})$. For this we work on finite level and use the commutative diagram

$$\begin{tikzcd} H^1(\mathcal{E}\setminus\mathcal{E}[c\ell^r],\mathbb{Z}/\ell^r\mathbb{Z}(1)) \arrow[r, "\mathrm{res}"] \arrow[d, equal, "\sim"] & H^0(\mathcal{E}[c\ell^r],\mathbb{Z}/\ell^r\mathbb{Z}) \arrow[d, equal, "\sim"] \\ H^1(\mathcal{E}\setminus\mathcal{E}[c],\mathcal{L}_{\mathbb{Z}/\ell^r\mathbb{Z}}(1)) \arrow[r] & H^0(\mathcal{E}[c],\mathcal{L}_{\mathbb{Z}/\ell^r\mathbb{Z}}). \end{tikzcd}$$

The residue of ${}_c\vartheta_{\mathcal{E}}$ is

$$c^2 1_{\{e\}}-1_{\mathcal{E}[c]}\in H^0(\mathcal{E}[c],\mathbb{Z}/\ell^r\mathbb{Z})\subset H^0(\mathcal{E}[c],\mathcal{L}_{\mathbb{Z}/\ell^r\mathbb{Z}})$$

and taking the limit shows that $\mathrm{comp}(\Theta_c)$ agrees with $\mathrm{pol}_{c^2 1_{\{e\}}-1_{\mathcal{E}[c]}}$ in $H^0(\mathcal{E}[c],\mathcal{L}og_{\mathbb{Q}_\ell}\mid_{\mathcal{E}[c]})$. $\square$

Corollary 8.5.3. *Let $S=Y(N)$, $t\in\mathcal{E}[N](S)$ be a non-zero N-torsion section and consider the k-th component $t^*\mathrm{comp}(\Theta_c)^k$ of*

$$t^*\mathrm{comp}(\Theta_c)\in\prod_{k\geq 0}H^1(Y(N),\mathrm{Sym}^k\mathscr{H}_{\mathbb{Q}_\ell}(1)).$$

Then one has

$$t^*\mathrm{comp}(\Theta_c)^k=\mathrm{mom}_t^k(\mathcal{E}\mathcal{S}_c^{\langle t\rangle})=t^*\mathrm{pol}_{c^2 1_{\{e\}}-1_{\mathcal{E}[c]}}=\frac{1}{N^k k!}\,{}_c\widetilde{e}_k(t).$$

In particular,

$$t^*\mathrm{comp}(\Theta_c)^k=\frac{-1}{N^{k-1}}(c^2\mathrm{Eis}^k_{\mathbb{Q}_\ell}(t)-c^{-k}\mathrm{Eis}^k_{\mathbb{Q}_\ell}([c]t))$$

and

$${}_c\widetilde{e}_k(t)=-Nk!(c^2\mathrm{Eis}^k_{\mathbb{Q}_\ell}(t)-c^{-k}\mathrm{Eis}^k_{\mathbb{Q}_\ell}([c]t)).$$

Proof. This is Theorem 8.5.2, Lemma 8.5.1 together with Lemma 8.4.3. $\square$

Corollary 8.5.4. *Let $S=Y(N)$ and consider the residue map from*

$$\mathrm{res}_\infty: H^1(Y(N),\mathrm{Sym}^k\mathscr{H}_{\mathbb{Q}_\ell}(1))\to H^0(\infty,\mathbb{Q}_\ell)\cong\mathbb{Q}_\ell,$$

then if $t=(a,b)\in(\mathbb{Z}/N\mathbb{Z})^2\setminus\{(0,0)\}$ one has

$$\mathrm{res}_\infty(\mathrm{Eis}^k_{\mathbb{Q}_\ell}(t))=\frac{-N^k}{(k+2)k!}B_{k+2}(\{\frac{a}{N}\}).$$

Proof. This is Corollary 8.5.3 together with Corollary 7.2.1 in the case $c \equiv 1 \mod N$. □

Remark 8.5.5. The formula differs by a minus sign from the one in [HK99] as we have a different uniformization of the elliptic curve.

9. The cup-product construction and its evaluation

9.1. **The cup-product construction.** Let $\pi : \mathcal{E} \to Y(N)$ be the universal elliptic curve. For a function $\psi : (\mathcal{E}[N] \setminus \{e\}) \to \mathbb{Q}_\ell$ we have the Eisenstein class

$$\mathrm{Eis}^k_{\mathbb{Q}_\ell}(\psi) \in H^1(Y(N), \mathrm{Sym}^k \mathscr{H}_{\mathbb{Q}_\ell}(1))$$

as in Definition 8.4.1. For $\phi, \psi : (\mathcal{E}[N] \setminus \{e\}) \to \mathbb{Q}_\ell$ we can consider the cup-product

$$\mathrm{Eis}^k_{\mathbb{Q}_\ell}(\phi) \cup \mathrm{Eis}^k_{\mathbb{Q}_\ell}(\psi) \in H^2(Y(N), \mathrm{Sym}^k \mathscr{H}_{\mathbb{Q}_\ell} \otimes \mathrm{Sym}^k \mathscr{H}_{\mathbb{Q}_\ell}(2)).$$

The Tate pairing $\mathscr{H}_{\mathbb{Q}_\ell} \otimes \mathscr{H}_{\mathbb{Q}_\ell} \to \mathbb{Q}_\ell(1)$ induces a pairing

$$\mathrm{Sym}^k \mathscr{H}_{\mathbb{Q}_\ell} \otimes \mathrm{Sym}^k \mathscr{H}_{\mathbb{Q}_\ell} \to \mathbb{Q}_\ell(k)$$

and we can consider

$$\mathrm{Eis}^k_{\mathbb{Q}_\ell}(\psi) \cup \mathrm{Eis}^k_{\mathbb{Q}_\ell}(\phi) \in H^2(Y(N), \mathbb{Q}_\ell(k+2)).$$

Let $\mathrm{res}_\infty : H^2(Y(N), \mathbb{Q}_\ell(k+2)) \to H^1(\infty, \mathbb{Q}_\ell(k+1))$ be the residue map at ∞, which is defined as in Definition 6.1.4 by the edge morphism of the Leray spectral sequence for Rj_*.

Definition 9.1.1. Let $\phi_\infty, \psi : (\mathcal{E}[N] \setminus \{e\}) \to \mathbb{Q}_\ell$ and suppose that

$$\mathrm{res}_\infty(\mathrm{Eis}^k_{\mathbb{Q}_\ell}(\phi_\infty)) = 1 \qquad \text{and} \qquad \mathrm{res}_\infty(\mathrm{Eis}^k_{\mathbb{Q}_\ell}(\psi)) = 0.$$

Then

$$\mathrm{Dir}(\psi) := \mathrm{res}_\infty(\mathrm{Eis}^k_{\mathbb{Q}_\ell}(\psi) \cup \mathrm{Eis}^k_{\mathbb{Q}_\ell}(\phi_\infty)) \in H^1(\infty, \mathbb{Q}_\ell(k+1))$$

is called the *cup-product construction* (compare [Hub, Definition 4.1.3.]).

In the next section we will prove the following theorem:

Theorem 9.1.2. *Let $\psi : (\mathcal{E}[N] \setminus \{e\}) \to \mathbb{Q}_\ell$ be a function such that*

$$\mathrm{res}_\infty(\mathrm{Eis}^k_{\mathbb{Q}_\ell}(\psi)) = 0$$

and identify $\mathcal{E}[N] \cong (\mathbb{Z}/N\mathbb{Z})^2$. Then one has

$$\mathrm{Dir}(\psi)) = \frac{-1}{Nk!} \sum_{b \in \mathbb{Z}/N\mathbb{Z} \setminus \{0\}} \psi(0,b) \widetilde{c}_{k+1}(\zeta_N^b) \in H^1(\infty, \mathbb{Q}_\ell(k+1)),$$

where $\widetilde{c}_{k+1}(\zeta_N^b)$ is the modified cyclotomic Soulé-Deligne element from Definition 4.1.3.

Recall commutative diagram of residue sequences

$$\begin{tikzcd} 0 \arrow[r] & H^1(\widetilde{Y}(N), j_*\mathrm{Sym}^k\mathscr{H}_{\mathbb{Q}_\ell}(1)) \arrow[r] \arrow[d] & H^1(Y(N), \mathrm{Sym}^k\mathscr{H}_{\mathbb{Q}_\ell}(1)) \arrow[r, "\mathrm{res}_\infty"] \arrow[d] & H^0(\infty, \mathbb{Q}_\ell) \arrow[d, equal] \\ 0 \arrow[r] & H^1(\widehat{X}(N)_\infty, j_*\mathrm{Sym}^k\mathscr{H}_{\mathbb{Q}_\ell}(1)) \arrow[r] & H^1(\widehat{Y}(N)_\infty, \mathrm{Sym}^k\mathscr{H}_{\mathbb{Q}_\ell}(1)) \arrow[r, "\mathrm{res}_\infty"] & H^0(\infty, \mathbb{Q}_\ell) \end{tikzcd}$$

from Corollary 6.1.5. For a function $\psi : (\mathcal{E}[N] \setminus \{e\}) \to \mathbb{Q}_\ell$ we have the Eisenstein class

$$\mathrm{Eis}^k_{\mathbb{Q}_\ell}(\psi) \in H^1(Y(N), \mathrm{Sym}^k\mathscr{H}_{\mathbb{Q}_\ell}(1))$$

as in Definition 8.4.1. Under the hypothesis $\mathrm{res}_\infty(\mathrm{Eis}^k_{\mathbb{Q}_\ell}(\psi)) = 0$, this can be considered as a class $\mathrm{Eis}^k_{\mathbb{Q}_\ell}(\psi) \in H^1(\widehat{X}(N)_\infty, j_*\mathrm{Sym}^k\mathscr{H}_{\mathbb{Q}_\ell}(1))$. For the proof of the following theorem we refer to [HK99, Theorem 2.1.4] or [Hub, Theorem 4.2.1].

Theorem 9.1.3 ([HK99])**.** *Assume that* $\mathrm{res}_\infty(\mathrm{Eis}^k_{\mathbb{Q}_\ell}(\psi)) = 0$ *so that* $\mathrm{Eis}^k_{\mathbb{Q}_\ell}(\psi) \in H^1(\widehat{X}(N)_\infty, j_*\mathrm{Sym}^k\mathscr{H}_{\mathbb{Q}_\ell}(1))$. *Then*

$$\mathrm{Dir}(\psi)) = \infty^*\mathrm{Eis}^k_{\mathbb{Q}_\ell}(\psi)$$

in $H^1(\infty, \infty^* j_*\mathrm{Sym}^k\mathscr{H}_{\mathbb{Q}_\ell}(1)) \cong H^1(\infty, \mathbb{Q}_\ell(k+1))$.

9.2. **Evaluation of the cup-product construction at infinity.** In this section we assume that $\mathrm{res}_\infty(\mathrm{Eis}^k_{\mathbb{Q}_\ell}(\psi)) = 0$ and are going to compute $\infty^*\mathrm{Eis}^k_{\mathbb{Q}_\ell}(\psi) \in H^1(\infty, \mathbb{Q}_\ell(k+1))$. The computation will take up the rest of this section.

In this section we will write $\widehat{Y} := \widehat{Y}(N)_\infty$ and $\widehat{X} := \widehat{X}(N)_\infty$ for brevity. Consider over $\widehat{Y}$ the map of sheaves

(9.2.1)

$$\mathrm{mult}_{\mathbb{Q}_\ell(1)} : \mathbb{Q}_\ell(1) \otimes \mathrm{Sym}^k\mathscr{H}_{\mathbb{Q}_\ell} \xrightarrow{\iota\otimes\mathrm{id}} \mathscr{H}_{\mathbb{Q}_\ell} \otimes \mathrm{Sym}^k\mathscr{H}_{\mathbb{Q}_\ell} \xrightarrow{\mathrm{mult}} \mathrm{Sym}^{k+1}\mathscr{H}_{\mathbb{Q}_\ell}.$$

We denote by $\mathrm{mult}_{\mathbb{Z}_\ell(1)}$ the same map with $\mathbb{Z}_\ell$-coefficients.

Lemma 9.2.1. *There is a commutative diagram*

$$\begin{tikzcd} H^1(\widehat{Y}, \mathrm{Sym}^k\mathscr{H}_{\mathbb{Q}_\ell}(1)) \arrow[r, "\mathrm{mult}_{\mathbb{Q}_\ell(1)}"] & H^1(\widehat{Y}, \mathrm{Sym}^{k+1}\mathscr{H}_{\mathbb{Q}_\ell}) \\ H^1(\widehat{X}, j_*\mathrm{Sym}^k\mathscr{H}_{\mathbb{Q}_\ell}(1)) \arrow[u, hook] \arrow[r, "\mathrm{mult}_{\mathbb{Q}_\ell(1)}"] \arrow[d, "\infty^*"] & H^1(\widehat{X}, j_*\mathrm{Sym}^{k+1}\mathscr{H}_{\mathbb{Q}_\ell}) \arrow[u, hook, "\cong"'] \arrow[d, "\infty^*"] \\ H^1(\infty, \mathbb{Q}_\ell(k+1)) \arrow[r, equal] & H^1(\infty, \mathbb{Q}_\ell(k+1)). \end{tikzcd}$$

In particular, to compute $\infty^*\mathrm{Eis}^k_{\mathbb{Q}_\ell}(\psi)$ *it suffices to compute the class of* $\infty^*\mathrm{mult}_{\mathbb{Q}_\ell(1)}\mathrm{Eis}^k_{\mathbb{Q}_\ell}(\psi)$.

Proof. It is clear that the upper square commutes and to show that the right upper vertical arrow is an isomorphism, note that $R^1 j_* \mathrm{Sym}^{k+1}\mathscr{H}_{\mathbb{Q}_\ell} \cong \infty_* \mathbb{Q}_\ell(-1)$ by Corollary 6.1.3 and that $H^0(\infty, \mathbb{Q}_\ell(-1)) = 0$. This, together with the Leray spectral sequence shows that the map

$$H^1(\widehat{X}, j_*\mathrm{Sym}^{k+1}\mathscr{H}_{\mathbb{Q}_\ell}) \to H^1(\widehat{Y}, \mathrm{Sym}^{k+1}\mathscr{H}_{\mathbb{Q}_\ell})$$

is an isomorphism. The pull-back of the map

$$\mathrm{mult}_{\mathbb{Q}_\ell(1)} : j_*\mathrm{Sym}^k\mathscr{H}_{\mathbb{Q}_\ell}(1) \to j_*(\mathrm{Sym}^{k+1}\mathscr{H}_{\mathbb{Q}_\ell})$$

via ∞^* gives $\infty^*\mathrm{mult}_{\mathbb{Q}_\ell(1)} : \mathbb{Q}_\ell(k+1) \to \mathbb{Q}_\ell(k+1)$, which, using Corollary 6.1.3, is just the identity and it follows that the lower square is commutative as well. $\square$

To use Lemma 9.2.1 for computations we recall from Corollary 8.5.3 that

$$\mathrm{mom}_t^k(\mathcal{ES}_c^{\langle t\rangle}) = \frac{-1}{N^{k-1}}(c^2\mathrm{Eis}^k_{\mathbb{Q}_\ell}(t) - c^{-k}\mathrm{Eis}^k_{\mathbb{Q}_\ell}([c]t)),$$

where $\mathrm{mom}_t^k : H^1(\widehat{Y}, \Lambda(\mathscr{H}_{\mathbb{Z}_\ell}\langle t\rangle)(1)) \to H^1(\widehat{Y}, \mathrm{Sym}^k\mathscr{H}_{\mathbb{Q}_\ell}(1))$ is the composition of mom_t^k with the canonical map (2.1.1). Taking this together with Lemma 9.2.1, we have to compute the image of $\mathcal{ES}_c^{\langle t\rangle}$ under the composition

(9.2.2)

$$H^1(\widehat{Y}, \Lambda(\mathscr{H}_{\mathbb{Z}_\ell}\langle t\rangle)(1)) \xrightarrow{\mathrm{mom}_t^k} H^1(\widehat{Y}, \mathrm{Sym}^k\mathscr{H}_{\mathbb{Z}_\ell}(1)) \xrightarrow{\mathrm{mult}_{\mathbb{Z}_\ell(1)}} H^1(\widehat{Y}, \mathrm{Sym}^{k+1}\mathscr{H}_{\mathbb{Z}_\ell}) \to$$
$$H^1(\widehat{Y}, \mathrm{Sym}^{k+1}\mathscr{H}_{\mathbb{Q}_\ell}) \xleftarrow{\cong} H^1(\widehat{X}, j_*\mathrm{Sym}^{k+1}\mathscr{H}_{\mathbb{Q}_\ell}) \xrightarrow{\infty^*} H^1(\infty, \mathbb{Q}_\ell(k+1)).$$

We will not compute this composition directly, but through some commutative diagrams on finite level, which will be stated below. To write them down, we need to introduce some notations. Let $I(\mathbb{Z}_\ell(1)) \subset \Lambda(\mathbb{Z}_\ell(1))$ and $I(\mathscr{H}_{\mathbb{Z}_\ell}) \subset \Lambda(\mathscr{H}_{\mathbb{Z}_\ell})$ be the augmentation ideals. Recall that $\Lambda(\mathscr{H}_{\mathbb{Z}_\ell}\langle t\rangle)$ is a $\Lambda(\mathscr{H}_{\mathbb{Z}_\ell})$-module and that the module structure is induced by the map $\mathscr{H}_{\mathbb{Z}_\ell} \times \mathscr{H}_{\mathbb{Z}_\ell}\langle t\rangle \to \mathscr{H}_{\mathbb{Z}_\ell}\langle t\rangle$, $(h, h') \mapsto h + h'$.

Definition 9.2.2. (1) We define ℓ-adic sheaves

$$\Lambda(\mathscr{H}_{\mathbb{Z}_\ell}\langle t\rangle)^{(n)} := \Lambda(\mathscr{H}_{\mathbb{Z}_\ell}\langle t\rangle)/I(\mathscr{H}_{\mathbb{Z}_\ell})^{n+1}\Lambda(\mathscr{H}_{\mathbb{Z}_\ell}\langle t\rangle)$$
$$\Lambda(\mathbb{Z}_\ell(1))^{(n)} := \Lambda(\mathbb{Z}_\ell(1))/I(\mathbb{Z}_\ell(1))^{n+1}$$

as the quotients by the $n+1$-power of the augmentation ideal.

(2) To have a shorter notation we let $\mathscr{H}_{\ell^r} := \mathscr{H}_{\mathbb{Z}/\ell^r\mathbb{Z}}$ and let

$$\Lambda_{\mathscr{H}_{\ell^r},t} := \mathbb{Z}/\ell^r\mathbb{Z}[\mathscr{H}_{\mathbb{Z}/\ell^r\mathbb{Z}}\langle t\rangle] \qquad \text{and} \qquad \Lambda_{\mu_{\ell^r},t} := \mathbb{Z}/\ell^r\mathbb{Z}[\mu_{\ell^r}\langle t\rangle]$$

where $\mu_{\ell^r}\langle t\rangle$ is the pre-image of $t \in \mathcal{E}[N]$ of the map

$$\mu_{\ell^r N} \xrightarrow{\iota} \mathcal{E}[\ell^r N] \to \mathcal{E}[N].$$

We write $\Lambda_{\mu_{\ell^r}}$ instead of $\Lambda_{\mu_{\ell^r},t}$ if $t = 0$ and let $I_{\mu_{\ell^r}} := \ker(\Lambda_{\mu_{\ell^r}} \to \mathbb{Z}/\ell^r\mathbb{Z})$ be the augmentation ideal.

(3) Consider the quotient maps $I(\mathbb{Z}_\ell(1)) \to I_{\mu_{\ell^r}}$ and $\Lambda(\mathscr{H}_{\mathbb{Z}_\ell}\langle t\rangle) \to \Lambda_{\mathscr{H}_{\ell^r},t}$, then we let $I^{(n)}_{\mu_{\ell^r}}$ and $\Lambda^{(n)}_{\mathscr{H}_{\ell^r},t}$ be the quotients of $I_{\mu_{\ell^r}}$ and $\Lambda_{\mathscr{H}_{\ell^r},t}$ by the images of the augmentation ideals $I(\mathbb{Z}_\ell(1))^{n+1}$ and $I(\mathscr{H}_{\mathbb{Z}_\ell}\langle t\rangle)^{n+1}$. Note that

$$I(\mathbb{Z}_\ell(1))^{(n)} = (I^{(n)}_{\mu_{\ell^r}})_{r\geq 1} \qquad \text{and} \qquad \Lambda(\mathscr{H}_{\mathbb{Z}_\ell}\langle t\rangle)^{(n)} = (\Lambda^{(n)}_{\mathscr{H}_{\ell^r},t})_{r\geq 1}$$

and that the canonical map $I(\mathbb{Z}_\ell(1))^{(n)} \to I(\mathbb{Z}_\ell(1))^{(1)} \cong \mathbb{Z}_\ell(1)$ induces $I^{(n)}_{\mu_{\ell^r}} \to \mathbb{Z}/\ell^r\mathbb{Z}(1)$.

We remark that $\Lambda(\mathscr{H}_{\mathbb{Z}_\ell}\langle t\rangle)^{(n)}$ and $I(\mathbb{Z}_\ell(1))^{(n)}$ are ℓ-adic sheaves. Comparing $\Lambda(\mathscr{H}_{\mathbb{Z}_\ell}\langle t\rangle)^{(n)}$ with the sheaf in Definition 8.2.3, we see that

$$\Lambda(\mathscr{H}_{\mathbb{Z}_\ell}\langle t\rangle)^{(n)} \cong t^*\mathcal{L}^{(n)}_{\mathbb{Z}_\ell}.$$

Over $\widehat{Y}$ we have an inclusion $\iota : \mathbb{Z}_\ell(1) \hookrightarrow \mathscr{H}_{\mathbb{Z}_\ell}$ and hence a map $\iota_! : \Lambda(\mathbb{Z}_\ell(1)) \to \Lambda(\mathscr{H}_{\mathbb{Z}_\ell})$, which when composed with the $\Lambda(\mathscr{H}_{\mathbb{Z}_\ell})$-module structure provides a homomorphism

$$\Lambda(\mathbb{Z}_\ell(1)) \otimes \Lambda(\mathscr{H}_{\mathbb{Z}_\ell}\langle t\rangle) \to \Lambda(\mathscr{H}_{\mathbb{Z}_\ell}\langle t\rangle).$$

It is clear that this homomorphism is compatible with the augmentation ideals and induces a homomorphism

$$I(\mathbb{Z}_\ell(1))^{(n)} \otimes \Lambda(\mathscr{H}_{\mathbb{Z}_\ell}\langle t\rangle)^{(n)} \to \Lambda(\mathscr{H}_{\mathbb{Z}_\ell}\langle t\rangle)^{(n)},$$

which has an obvious analogue on finite levels

$$I^{(n)}_{\mu_{\ell^r}} \otimes \Lambda^{(n)}_{\mathscr{H}_{\ell^r},t} \to \Lambda^{(n)}_{\mathscr{H}_{\ell^r},t}. \tag{9.2.3}$$

With these notations we define a map

$$\kappa : H^1(\widehat{Y}, \Lambda_{\mathscr{H}_{\ell^r},t}(1)) \to H^1(\widehat{Y}, \underline{\mathrm{Hom}}_{\widehat{Y}}(I^{(n)}_{\mu_{\ell^r}}, \mathrm{Sym}^{k+1}\mathscr{H}_{\ell^r}(1))) \tag{9.2.4}$$

as the composition

$$\kappa : H^1(\widehat{Y}, \Lambda_{\mathscr{H}_{\ell^r},t}(1)) \xrightarrow{\otimes I^{(n)}_{\mu_{\ell^r}}} H^1(\widehat{Y}, \underline{\mathrm{Hom}}_{\widehat{Y}}(I^{(n)}_{\mu_{\ell^r}}, I^{(n)}_{\mu_{\ell^r}} \otimes \Lambda^{(n)}_{\mathscr{H}_{\ell^r},t}(1))) \xrightarrow{9.2.3}$$
$$H^1(\widehat{Y}, \underline{\mathrm{Hom}}_{\widehat{Y}}(I^{(n)}_{\mu_{\ell^r}}, \Lambda^{(n)}_{\mathscr{H}_{\ell^r},t}(1))) \xrightarrow{\mathrm{mom}_t^{k+1}} H^1(\widehat{Y}, \underline{\mathrm{Hom}}_{\widehat{Y}}(I^{(n)}_{\mu_{\ell^r}}, \mathrm{Sym}^{k+1}\mathscr{H}_{\ell^r}(1))).$$

We can now write down the first commutative diagram:

Lemma 9.2.3. *With the map κ defined in* (9.2.4), *there is a commutative diagram*

$$\begin{tikzcd}
H^1(\widehat{Y}, \Lambda_{\mathscr{H}_{\ell^r},t}(1)) \arrow[r, "\mathrm{mom}_t^k"] \arrow[d, "\kappa"'] & H^1(\widehat{Y}, \mathrm{Sym}^k \mathscr{H}_{\ell^r}(1)) \arrow[d, "\mathrm{mult}_{\mathbb{Z}/\ell^r\mathbb{Z}(1)}"] \\
H^1(\widehat{Y}, \underline{\mathrm{Hom}}_{\widehat{Y}}(I^{(n)}_{\mu_{\ell^r}}, \mathrm{Sym}^{k+1}\mathscr{H}_{\ell^r}(1))) & H^1(\widehat{Y}, \mathrm{Sym}^{k+1}\mathscr{H}_{\ell^r}) \arrow[l, "(a)"'] \\
H^1(\widehat{X}, \underline{\mathrm{Hom}}_{\widehat{X}}(I^{(n)}_{\mu_{\ell^r}}, j_*\mathrm{Sym}^{k+1}\mathscr{H}_{\ell^r}(1))) \arrow[u, "(b)"] \arrow[d, "\infty^*"'] & H^1(\widehat{X}, j_*\mathrm{Sym}^{k+1}\mathscr{H}_{\ell^r}) \arrow[l, "(a)"'] \arrow[u, "(b)"'] \arrow[d, "\infty^*"] \\
H^1(\infty, \underline{\mathrm{Hom}}_{\infty}(I^{(n)}_{\mu_{\ell^r}}, \mathbb{Z}/\ell^r\mathbb{Z}(k+2))) & H^1(\infty, \mathbb{Z}/\ell^r\mathbb{Z}(k+1)). \arrow[l, "(a)"']
\end{tikzcd}$$

The maps (a) *are induced from*

$$\mathrm{Sym}^{k+1}\mathscr{H}_{\ell^r} \cong \underline{\mathrm{Hom}}_{\widehat{Y}}(\mathbb{Z}/\ell^r\mathbb{Z}(1), \mathrm{Sym}^{k+1}\mathscr{H}_{\ell^r}(1)) \to \underline{\mathrm{Hom}}_{\widehat{Y}}(I^{(n)}_{\mu_{\ell^r}}, \mathrm{Sym}^{k+1}\mathscr{H}_{\ell^r}(1)),$$

where the last map is pull-back by $I^{(n)}_{\mu_{\ell^r}} \to \mathbb{Z}/\ell^r\mathbb{Z}(1)$. *The map* (b) *is induced by* $j_*\mathrm{Sym}^{k+1}\mathscr{H}_{\ell^r} \to Rj_*\mathrm{Sym}^{k+1}\mathscr{H}_{\ell^r}$ *and in the last line the isomorphism* $\infty^* j_*\mathrm{Sym}^{k+1}\mathscr{H}_{\ell^r} \cong \mathbb{Z}/\ell^r\mathbb{Z}(k+1)$ *is used.*

Proof. The commutativity of the two lower squares is clear from the definition of the map (a). For the upper square note that the diagram

$$\begin{tikzcd}
I^{(n)}_{\mu_{\ell^r}} \otimes \Lambda^{(n)}_{\mathscr{H}_{\ell^r},t} \arrow[r, "(9.2.3)"] \arrow[d, "\mathrm{mom}_t^k"'] & \Lambda^{(n)}_{\mathscr{H}_{\ell^r},t} \arrow[d, "\mathrm{mom}_t^{k+1}"] \\
\mathrm{Sym}^k\mathscr{H}_{\ell^r}(1) \arrow[r, "\mathrm{mult}_{\mathbb{Z}/\ell^r\mathbb{Z}(1)}"] & \mathrm{Sym}^{k+1}\mathscr{H}_{\ell^r}
\end{tikzcd}$$

commutes. This gives the commutative diagram

$$\begin{tikzcd}
\underline{\mathrm{Hom}}_{\widehat{Y}}(I^{(n)}_{\mu_{\ell^r}}, I^{(n)}_{\mu_{\ell^r}} \otimes \Lambda^{(n)}_{\mathscr{H}_{\ell^r},t}(1)) \arrow[r, "\mathrm{mom}_t^k"] \arrow[d, "(9.2.3)"'] & \underline{\mathrm{Hom}}_{\widehat{Y}}(I^{(n)}_{\mu_{\ell^r}}, \mathrm{Sym}^k\mathscr{H}_{\ell^r}(1)) \arrow[d, "\mathrm{mult}_{\mathbb{Z}/\ell^r\mathbb{Z}(1)}"] \\
\underline{\mathrm{Hom}}_{\widehat{Y}}(I^{(n)}_{\mu_{\ell^r}}, \Lambda^{(n)}_{\mathscr{H}_{\ell^r},t}) \arrow[r, "\mathrm{mom}_t^{k+1}"] & \underline{\mathrm{Hom}}_{\widehat{Y}}(I^{(n)}_{\mu_{\ell^r}}, \mathrm{Sym}^{k+1}\mathscr{H}_{\ell^r}).
\end{tikzcd}$$

Furthermore, the canonical map $I^{(n)}_{\mu_{\ell^r}} \to \mathbb{Z}/\ell^r\mathbb{Z}(1)$ induces a commutative diagram

$$\begin{tikzcd}
\underline{\mathrm{Hom}}_{\widehat{Y}}(I^{(n)}_{\mu_{\ell^r}}, \mathrm{Sym}^k\mathscr{H}_{\ell^r}(1)) \arrow[d, "\mathrm{mult}_{\mathbb{Z}/\ell^r\mathbb{Z}(1)}"] & \underline{\mathrm{Hom}}_{\widehat{Y}}(\mathbb{Z}/\ell^r\mathbb{Z}(1), \mathrm{Sym}^k\mathscr{H}_{\ell^r}(1)) \arrow[l] \arrow[d, "\mathrm{mult}_{\mathbb{Z}/\ell^r\mathbb{Z}(1)}"] \\
\underline{\mathrm{Hom}}_{\widehat{Y}}(I^{(n)}_{\mu_{\ell^r}}, \mathrm{Sym}^{k+1}\mathscr{H}_{\ell^r}) & \underline{\mathrm{Hom}}_{\widehat{Y}}(\mathbb{Z}/\ell^r\mathbb{Z}(1), \mathrm{Sym}^{k+1}\mathscr{H}_{\ell^r}) \arrow[l]
\end{tikzcd}$$

Together, the commutativity of the upper square follows. $\square$

The next lemma explains that in the commutative diagram in Lemma 9.2.3 we can use the left hand column to evaluate the elliptic Soulé elements.

Lemma 9.2.4. *The map of $\mathbb{Q}_\ell$-sheaves*

$$I(\mathbb{Z}_\ell(1))^{(n)}_{\mathbb{Q}_\ell} \to \mathbb{Q}_\ell(1)$$

obtained as the limit of the canonical maps $I^{(n)}_{\mu_{\ell^r}} \to \mathbb{Z}/\ell^r\mathbb{Z}(1)$ is split. In particular, the maps

$$H^1(\widehat{Y}, \mathrm{Sym}^{k+1}\mathscr{H}_{\mathbb{Q}_\ell}) \to H^1(\widehat{Y}, \underline{\mathrm{Hom}}_{\widehat{Y}}(I(\mathbb{Z}_\ell(1))^{(n)}_{\mathbb{Q}_\ell}, \mathrm{Sym}^{k+1}\mathscr{H}_{\mathbb{Q}_\ell}(1)))$$
$$H^1(\infty, \mathbb{Q}_\ell(k+1)) \to H^1(\infty, \underline{\mathrm{Hom}}_\infty(I(\mathbb{Z}_\ell(1))^{(n)}_{\mathbb{Q}_\ell}, \mathbb{Q}_\ell(k+2)))$$

obtained from taking the limits of the maps (1) *in Lemma 9.2.3 have splittings.*

Proof. This follows immediately from the fact that $I(\mathbb{Z}_\ell(1))^{(n)}_{\mathbb{Q}_\ell} \cong \bigoplus_{k=1}^n \mathbb{Q}_\ell(k)$. $\square$

From Lemmas 9.2.3 and 9.2.4 one sees that to compute the image of $\mathcal{ES}_c^{\langle t\rangle}$ under the map (9.2.2), it is sufficient to compute the image under the left column in Lemma 9.2.3 and to apply the splitting from Lemma 9.2.4 in the limit.

To do this, we interpret this left column in terms of functions on $\mu_{\ell^r} \times \mathcal{E}[\ell^r]\langle t\rangle$. Let $q_r\langle t\rangle : \mu_{\ell^r} \to \widehat{Y}$ and $p_r\langle t\rangle : \mathcal{E}[\ell^r]\rangle t\langle \to \widehat{Y}$ be the structure maps. We define

$$\mathbb{Z}_{\mu_{\ell^r},t} := q_r\langle t\rangle_*\mathbb{Z} \qquad \text{and} \qquad \mathbb{Z}_{\mathscr{H}_{\ell^r},t} := p_r\langle t\rangle_*\mathbb{Z} \tag{9.2.5}$$

and write $\mathbb{Z}_{\mu_{\ell^r}} := \mathbb{Z}_{\mu_{\ell^r},t}$ in the case $t = 0$. Let $\mathbb{Z}^0_{\mu_{\ell^r}} := \ker(\mathbb{Z}_{\mu_{\ell^r}} \to \mathbb{Z})$ the kernel of the augmentation. As in (9.2.3), the translation action of μ_{ℓ^r} on $\mathcal{E}[\ell^r]\langle t\rangle$ induces a map of sheaves on $\widehat{Y}$

$$\mathbb{Z}^0_{\mu_{\ell^r}} \otimes \mathbb{G}_m \to \mathbb{G}_m, \tag{9.2.6}$$

which is induced on T-valued points by $(\delta_\alpha - \delta_1) \otimes f \mapsto \frac{\tau_{\alpha,*}f}{f}$, where τ_α is the translation by $\iota(\alpha)$.

Lemma 9.2.5. *The diagram*

$$\begin{tikzcd}
\mathrm{Hom}_{\widehat{Y}}(\mathbb{Z}_{\mathscr{H}_{\ell^r,t}}, \mathbb{G}_m) \arrow[r, "\partial_r"] \arrow[d, "(1)"'] & H^1(\widehat{Y}, \Lambda_{\mathscr{H}_{\ell^r,t}}(1)) \arrow[d, "\kappa"] \\
\mathrm{Hom}_{\widehat{Y}}(\mathbb{Z}^0_{\mu_{\ell^r}} \otimes \mathbb{Z}_{\mathscr{H}_{\ell^r,t}}, \mathbb{G}_m) \arrow[r, "(2)"] & H^1(\widehat{Y}, \underline{\mathrm{Hom}}_{\widehat{Y}}(I^{(n)}_{\mu_{\ell^r}}, \mathrm{Sym}^{k+1}\mathscr{H}_{\ell^r}(1))) \\
\mathrm{Hom}_{\widehat{X}}(\mathbb{Z}^0_{\mu_{\ell^r}} \otimes j_*\mathbb{Z}_{\mathscr{H}_{\ell^r,t}}, \mathbb{G}_m) \arrow[r, "(3)"] \arrow[u, hook, "j^*"] \arrow[d, "\infty^*"'] & H^1(\widehat{X}, \underline{\mathrm{Hom}}_{\widehat{X}}(I^{(n)}_{\mu_{\ell^r}}, j_*\mathrm{Sym}^{k+1}\mathscr{H}_{\ell^r}(1))) \arrow[u, hook, "j^*"'] \arrow[d, "\infty^*"] \\
\mathrm{Hom}_{\infty}(\mathbb{Z}^0_{\mu_{\ell^r}} \otimes \infty^* j_*\mathbb{Z}_{\mathscr{H}_{\ell^r,t}}, \mathbb{G}_m) \arrow[r, "(4)"] \arrow[d, "(5)"'] & H^1(\infty, \underline{\mathrm{Hom}}_{\infty}(I^{(n)}_{\mu_{\ell^r}}, \mathbb{Z}/\ell^r\mathbb{Z}(k+2))) \\
\mathrm{Hom}_{\infty}(\mathbb{Z}^0_{\mu_{\ell^r}} \otimes \mathbb{Z}_{\mu_{\ell^r,t}}, \mathbb{G}_m), \arrow[ur, "\varrho"'] &
\end{tikzcd}$$

where the labelled maps are defined in the proof, commutes.

Proof. First consider the upper square. There is a commutative diagram

$$\begin{tikzcd}
\underline{\mathrm{Hom}}(\mathbb{Z}_{\mathscr{H}_{\ell^r,t}}, \mu_{\ell^r}) \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}(\mathbb{Z}_{\mathscr{H}_{\ell^r,t}}, \mathbb{G}_m) \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}(\mathbb{Z}_{\mathscr{H}_{\ell^r,t}}, \mathbb{G}_m) \arrow[d] \\
\underline{\mathrm{Hom}}(\mathbb{Z}^0_{\mu_{\ell^r}} \otimes \mathbb{Z}_{\mathscr{H}_{\ell^r,t}}, \mu_{\ell^r}) \arrow[r] & \underline{\mathrm{Hom}}(\mathbb{Z}^0_{\mu_{\ell^r}} \otimes \mathbb{Z}_{\mathscr{H}_{\ell^r,t}}, \mathbb{G}_m) \arrow[r] & \underline{\mathrm{Hom}}(\mathbb{Z}^0_{\mu_{\ell^r}} \otimes \mathbb{Z}_{\mathscr{H}_{\ell^r,t}}, \mathbb{G}_m)
\end{tikzcd}$$

of Kummer sequences, where the vertical arrows are obtained by tensoring with $\mathbb{Z}^0_{\mu_{\ell^r}}$ and composing with the map (9.2.6). If we use the fact that $\Lambda_{\mathscr{H}_{\ell^r,t}} = p_r\langle t\rangle_*\mathbb{Z}/\ell^r\mathbb{Z}$ is selfdual, because $p_r\langle t\rangle$ is finite, we get

$$\underline{\mathrm{Hom}}(\mathbb{Z}^0_{\mu_{\ell^r}} \otimes \mathbb{Z}_{\mathscr{H}_{\ell^r,t}}, \mu_{\ell^r}) = \underline{\mathrm{Hom}}(I_{\mu_{\ell^r}} \otimes \Lambda_{\mathscr{H}_{\ell^r,t}}, \mu_{\ell^r}) \to \underline{\mathrm{Hom}}(I^{(n)}_{\mu_{\ell^r}}, \Lambda^{(n)}_{\mathscr{H}_{\ell^r,t}}(1)).$$

This induces the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\widehat{Y}}(\mathbb{Z}_{\mathscr{H}_{\ell^r,t}}, \mathbb{G}_m) \arrow[r, "\partial_r"] \arrow[d, "(9.2.6)"'] & H^1(\widehat{Y}, \Lambda_{\mathscr{H}_{\ell^r,t}}(1)) \arrow[d, "(9.2.3)"] \\
\mathrm{Hom}_{\widehat{Y}}(\mathbb{Z}^0_{\mu_{\ell^r}} \otimes \mathbb{Z}_{\mathscr{H}_{\ell^r,t}}, \mathbb{G}_m) \arrow[r] & H^1(\widehat{Y}, \underline{\mathrm{Hom}}_{\widehat{Y}}(I^{(n)}_{\mu_{\ell^r}}, \Lambda^{(n)}_{\mathscr{H}_{\ell^r,t}}(1))).
\end{tikzcd}$$

Composition with mom_t^{k+1} of the lower horizontal map gives the map (2) and shows that the upper square commutes. Next consider the square in the middle. The map (3) is obtained in a similar way as the map (2). One considers the above commutative diagram of Kummer sequences but now with $\mathbb{Z}_{\mathscr{H}_{\ell^r,t}}$ replaced by $j_*\mathbb{Z}_{\mathscr{H}_{\ell^r,t}}$. Furthermore, one uses the fact that $j_*\Lambda_{\mathscr{H}_{\ell^r,t}}$ is still selfdual because $\widehat{X}$ is regular of dimension one (see [Del77,

Dualité, Théorème 1.3]). From this definition of the map (3) it is clear that the diagram in the middle commutes. Finally, the map (4) is again defined as the map (2). First one uses the inclusion $\mathbb{Z}_{\mu_{\ell^r},t} \subset \infty^* j_* \mathbb{Z}_{\mathscr{H}_{\ell^r},t}$, which induces the map (5). Then one considers the commutative diagram of Kummer sequences with $\mathbb{Z}_{\mu_{\ell^r},t}$ instead of $\mathbb{Z}_{\mathscr{H}_{\ell^r},t}$. The rest of the definition is as above and it is also clear that the lower square commutes with these definitions. □

Write $K := \mathbb{Q}(\zeta_N)((q^{1/N}))$ and fix an algebraic closure $\overline{K}$ of K and $\overline{\mathbb{Q}}$ of $\mathbb{Q}$. Let G_K and $G_{\mathbb{Q}}$ the associated absolute Galois groups. Then we can identify

$$\mathrm{Hom}_{\widehat{Y}}(\mathbb{Z}_{\mathscr{H}_{\ell^r},t}, \mathbb{G}_m) = \mathrm{Hom}(\mathbb{Z}[\mathcal{E}[\ell^r]\langle t\rangle(\overline{K})], \overline{K}^*)^{G_K}$$

with the set of maps from $\mathcal{E}[\ell^r]\langle t\rangle(\overline{K})$ to $\overline{K}^*$. Similarly,

$$\mathrm{Hom}_\infty(\mathbb{Z}^0_{\mu_{\ell^r}} \otimes \mathbb{Z}_{\mu_{\ell^r},t}, \mathbb{G}_m) = \mathrm{Hom}(\mathbb{Z}[\mu_{\ell^r}(\overline{\mathbb{Q}})]^0 \otimes \mathbb{Z}[\mu_{\ell^r}\langle t\rangle(\overline{\mathbb{Q}})], \overline{\mathbb{Q}}^*)^{G_{\mathbb{Q}}}.$$

With this notation we have:

Proposition 9.2.6. *Let $t = (a,b) \in (\mathbb{Z}/N\mathbb{Z})^2 \cong \mathcal{E}[N]$ and $t \neq (0,0)$. Then the image of ${}_c\vartheta_{\mathcal{E}} \in \mathrm{Hom}_{\widehat{Y}}(\mathbb{Z}_{\mathscr{H}_{\ell^r},t}, \mathbb{G}_m)$ in $\mathrm{Hom}_{\widehat{Y}}(\mathbb{Z}^0_{\mu_{\ell^r}} \otimes \mathbb{Z}_{\mathscr{H}_{\ell^r},t}, \mathbb{G}_m)$ under the map* (1) *from Lemma 9.2.5 is the restriction j^* of a function ${}_cF_r$ and the image of ${}_cF_r$ under ∞^* and* (5) *in Lemma 9.2.5 is given by*

$${}_cF_r{}^\infty := (5) \circ \infty^* {}_cF_r : (\delta_\alpha - \delta_1) \otimes \beta \mapsto \begin{cases} \alpha^{\frac{c^2-c}{2}} & a = 0 \\ \alpha^{\frac{c^2-c}{2}} \left(\frac{1-\beta\alpha}{1-\beta}\right)^{c^2} \left(\frac{1-\beta^c}{1-\beta^c\alpha^c}\right) & a \neq 0 \end{cases}$$

where $\beta \in \mu_{\ell^r}\langle t\rangle(\overline{\mathbb{Q}})$ and $(\delta_\alpha - \delta_1) \in \mathbb{Z}[\mu_{\ell^r}(\overline{\mathbb{Q}})]^0$.

Proof. The image of ${}_c\vartheta_{\mathcal{E}}$ under (1) is the function $(\delta_\alpha - \delta_1) \otimes Q \mapsto \frac{{}_c\vartheta_{\mathcal{E}}(Q+\iota(\alpha))}{{}_c\vartheta_{\mathcal{E}}(Q)}$, where $Q \in \mathcal{E}[\ell^r]\langle t\rangle$. The computation in Theorem 7.1.3 shows that the element

$$\frac{{}_c\vartheta_{\mathcal{E}}(Q+\iota(\alpha))}{{}_c\vartheta_{\mathcal{E}}(Q)} \in \overline{\mathbb{Q}(\zeta_N)((q^{1/N}))}^*$$

has no residue at ∞, hence lies in $\overline{\mathbb{Q}(\zeta_N)[[q^{1/N}]]}$ and is in the image of the map j^*. To take the image of this element under ∞^* and (5) means to take the leading term of the power series $\frac{{}_c\vartheta_{\mathcal{E}}(Q+\iota(\alpha))}{{}_c\vartheta_{\mathcal{E}}(Q)}$ and to restrict the Q's to $\iota(\mu_{\ell^r}\langle t\rangle(\overline{\mathbb{Q}}))$. To evaluate the function $\frac{{}_c\vartheta_{\mathcal{E}}(Q+\iota(\alpha))}{{}_c\vartheta_{\mathcal{E}}(Q)}$ we look at the formula in Corollary 5.1.2 for $Q = \frac{x\tau}{\ell^r N} + \frac{y}{\ell^r N}$. This gives

$$\begin{aligned} \frac{{}_c\vartheta(Q)}{{}_c\vartheta(Q+\iota(\alpha))} =& \alpha^{\frac{c^2-c}{2}} \left(\frac{1-q_\tau^{\frac{x}{\ell^r N}}\zeta_{\ell^r N}^y\alpha}{1-q_\tau^{\frac{x}{\ell^r N}}\zeta_{\ell^r N}^y}\right)^{c^2} \left(\frac{1-q_\tau^{\frac{cx}{\ell^r N}}\zeta_{\ell^r N}^{cy}}{1-q_\tau^{\frac{cx}{\ell^r N}}\zeta_{\ell^r N}^{cy}\alpha^c}\right) \\ &\times \left(\frac{\widetilde{\gamma}_{q_\tau}(q_\tau^{\frac{x}{\ell^r N}}\zeta_{\ell^r N}^y\alpha)}{\widetilde{\gamma}_{q_\tau}(q_\tau^{\frac{x}{\ell^r N}}\zeta_{\ell^r N}^y)}\right)^{c^2} \left(\frac{\widetilde{\gamma}_{q_\tau}(q_\tau^{\frac{cx}{\ell^r N}}\zeta_{\ell^r N}^{cy})}{\widetilde{\gamma}_{q_\tau}(q_\tau^{\frac{cx}{\ell^r N}}\zeta_{\ell^r N}^{cy}\alpha^c)}\right). \end{aligned}$$

If $x \neq 0$ one sees directly from this formula (as q_τ^x then goes to zero if q_τ goes to zero) that the leading term is $\alpha^{\frac{c^2-c}{2}}$. If $x = 0$ the leading term for $Q = \beta = \zeta_{\ell^r N}^y$ is

$$\alpha^{\frac{c^2-c}{2}} \left(\frac{1-\beta\alpha}{1-\beta}\right)^{c^2} \left(\frac{1-\beta^c}{1-\beta^c\alpha^c}\right).$$

This is the desired result. $\square$

Lemma 9.2.7. *Consider the function*

$$G_r : (\delta_\alpha - \delta_1) \otimes \beta \mapsto \alpha^{\frac{c^2-c}{2}}$$

in $\mathrm{Hom}_\infty(\mathbb{Z}^0_{\mu_{\ell^r}} \otimes \mathbb{Z}_{\mu_{\ell^r},t}, \mathbb{G}_m)$ *and its image under the map* ϱ *of Lemma 9.2.5 in*

$$H^1(\infty, \underline{\mathrm{Hom}}_\infty(I^{(n)}_{\mu_{\ell^r}}, \Lambda_{\mu_{\ell^r},t}(1))).$$

Then $\varprojlim_r \varrho(G_r) = 0$ *in* $H^1(\infty, \underline{\mathrm{Hom}}_\infty(I(\mathbb{Z}_\ell(1))^{(n)}, \Lambda(\mathbb{Z}_\ell(1)\langle t\rangle)(1))$. *In particular,* $\varprojlim_r \varrho({}_cF_r^\infty)$ *and* $\varprojlim_r \varrho({}_cF_r{}^\infty G_r^{-1})$ *are equal in*

$$H^1(\infty, \underline{\mathrm{Hom}}_\infty(I(\mathbb{Z}_\ell(1))^{(n)}, \Lambda(\mathbb{Z}_\ell(1)\langle t\rangle)(1)).$$

Proof. The function F corresponds to the canonical inclusion in $\mathrm{Hom}_\infty(\mu_{\ell^r}, \mathbb{G}_m)$ under the map

$$\mathrm{Hom}_\infty(\mu_{\ell^r}, \mathbb{G}_m) \to \mathrm{Hom}_\infty(\mathbb{Z}^0_{\mu_{\ell^r}} \otimes \mathbb{Z}_{\mu_{\ell^r},t}, \mathbb{G}_m)$$

induced by the surjections $\mathbb{Z}^0_{\mu_{\ell^r}} \to \mu_{\ell^r}$ and $\mathbb{Z}_{\mu_{\ell^r},t} \to \mathbb{Z}$. From this it follows that $\varrho(G_r)$ is in the image of

$$H^1(\infty, \underline{\mathrm{Hom}}_\infty(\mu_{\ell^r}, \mathbb{Z}/\ell^r\mathbb{Z}(1)) \to H^1(\infty, \underline{\mathrm{Hom}}_\infty(I^{(n)}_{\mu_{\ell^r}}, \Lambda_{\mu_{\ell^r},t}(1))) \tag{9.2.7}$$

which is induced by $I^{(n)}_{\mu_{\ell^r}} \to \mu_{\ell^r}$ and $\mathbb{Z}/\ell^r\mathbb{Z}(1) \to \Lambda_{\mu_{\ell^r},t}(1)$ corresponding to the canonical map $\mathbb{Z}/\ell^r\mathbb{Z}(1) \to p_r\langle t\rangle_*\mathbb{Z}/\ell^r\mathbb{Z}(1)$. But the map in (9.2.7) factors through

$$H^1(\infty, \underline{\mathrm{Hom}}_\infty(\mu_{\ell^r}, \Lambda_{\mu_{\ell^r},t}(1))) = H^1(\infty, \Lambda_{\mu_{\ell^r},t}) = H^1(\mu_{\ell^r,t}, \mathbb{Z}/\ell^r\mathbb{Z}(1))$$

so that we get from (9.2.7)

$$H^1(\infty, \mathbb{Z}/\ell^r\mathbb{Z}) \xrightarrow{p_r\langle t\rangle^*} H^1(\mu_{\ell^r,t}, \mathbb{Z}/\ell^r\mathbb{Z}(1)).$$

We have to show that the image of the canonical class in $H^1(\infty, \mathbb{Z}/\ell^r\mathbb{Z})$ under $p_r\langle t\rangle^*$ vanishes in the limit. As the corestriction

$$H^1(\mu_{\ell^r,t}, \mathbb{Z}/\ell^r\mathbb{Z}(1)) \to H^1(\mu_{\ell^{r-1},t}, \mathbb{Z}/\ell^{r-1}\mathbb{Z}(1))$$

multiplies this class by ℓ, the result follows. $\square$

The next commutative diagram allows to relate this computation to the moments of $\mathcal{S}^{\langle\zeta_N^b\rangle}_{c,r}$.

Lemma 9.2.8. *The diagram*

$$\begin{tikzcd}
\mathrm{Hom}_\infty(\mathbb{Z}_{\mu_{\ell^r},t}, \mathbb{G}_m) \arrow[r, "\partial_r"] \arrow[d, "(1)"'] & H^1(\infty, \Lambda_{\mu_{\ell^r},t}(1)) \arrow[r, "\mathrm{mom}_t^k"] \arrow[d, "\kappa"] & H^1(\infty, \mathbb{Z}/\ell^r\mathbb{Z}(k+1)) \arrow[dl, "(*)"] \\
\mathrm{Hom}_\infty(\mathbb{Z}^0_{\mu_{\ell^r}} \otimes \mathbb{Z}_{\mu_{\ell^r},t}, \mathbb{G}_m) \arrow[r, "\varrho"] & H^1(\infty, \underline{\mathrm{Hom}}_\infty(I^{(n)}_{\mu_{\ell^r}}, \mathbb{Z}/\ell^r\mathbb{Z}(k+2))) &
\end{tikzcd}$$

commutes. Here the maps κ, (1) and ϱ are defined exactly as in the proof of Lemma 9.2.5 and the map $(*)$ is induced by the canonical map $I^{(n)}_{\mu_{\ell^r}} \to \mathbb{Z}/\ell^r\mathbb{Z}(1)$, which gives

$$\mathbb{Z}/\ell^r\mathbb{Z}(k+1) \cong \underline{\mathrm{Hom}}_\infty(\mathbb{Z}/\ell^r\mathbb{Z}(1), \mathbb{Z}/\ell^r\mathbb{Z}(k+2)) \to \underline{\mathrm{Hom}}_\infty(I^{(n)}_{\mu_{\ell^r}}, \mathbb{Z}/\ell^r\mathbb{Z}(k+2)).$$

Proof. As in the proof of Lemma 9.2.5. □

Lemma 9.2.9. *Consider the function ${}_c\Xi$ from (4.2.1). Its image under the map (1) in Lemma 9.2.8 is the function ${}_cF_r{}^\infty G_r^{-1}$*

$$(\delta_\alpha - \delta_1) \otimes \beta \mapsto \left(\frac{1-\beta\alpha}{1-\beta}\right)^{c^2} \left(\frac{1-\beta^c}{1-\beta^c\alpha^c}\right)$$

Proof. Clear from the definition. □

Lemma 9.2.10. *Let $t = (a,b) \in (\mathbb{Z}/N\mathbb{Z})^2 \cong \mathcal{E}[N]$ and $t \neq (0,0)$. The image of*

$$\mathcal{S}_c^{\langle \zeta_N^b \rangle} \in H^1(\infty, \Lambda(\mathbb{Z}_\ell(1)\langle t \rangle)(1))$$

under κ coincides with the image of

$$\mathcal{E}\mathcal{S}_c^{\langle t \rangle} \in H^1(\widehat{Y}, \Lambda(\mathscr{H}_{\mathbb{Z}_\ell}\langle t \rangle)(1))$$

under the right hand column in Lemma 9.2.5 in

$$H^1(\infty, \underline{\mathrm{Hom}}_\infty(I(\mathbb{Z}_\ell(1))^{(n)}, \mathbb{Z}_\ell(k+2)).$$

Proof. Follows by putting together Lemmas 9.2.5, 9.2.7, 9.2.8 and Proposition 9.2.6. □

The next lemma shows that the image in $H^1(\infty, \underline{\mathrm{Hom}}_\infty(I(\mathbb{Z}_\ell(1))^{(n)}, \mathbb{Z}_\ell(k+2))$ determines the element in $H^1(\infty, \mathbb{Q}_\ell(k+1))$.

Lemma 9.2.11. *The map of $\mathbb{Q}_\ell$ vector spaces*

$$H^1(\infty, \mathbb{Q}_\ell(k+1)) \to H^1(\infty, \underline{\mathrm{Hom}}_\infty(I(\mathbb{Z}_\ell(1))_{\mathbb{Q}_\ell}, \mathbb{Q}_\ell(k+2)))$$

induced by the inverse limit of the map $(*)$ in Lemma 9.2.8 has a splitting.

Proof. This follows again from the fact that $I(\mathbb{Z}_\ell(1))^{(n)}_{\mathbb{Q}_\ell} \cong \bigoplus_{k=1}^n \mathbb{Q}_\ell(k)$. □

We are ready to give the proof of Theorem 9.1.2.

Proof of Theorem 9.1.2. Let $\mathrm{Eis}^k_{\mathbb{Q}_\ell}(\psi) \in H^1(\widehat{X}, j_*\mathrm{Sym}^k \mathscr{H}_{\mathbb{Q}_\ell}(1))$ be as in the Theorem. By Lemma 9.2.1 it is enough to compute $\infty^*\mathrm{mult}_{\mathbb{Q}_\ell(1)}(\mathrm{Eis}^k_{\mathbb{Q}_\ell}(\psi))$. To do this note that for $c \equiv 1 \mod N$ by Corollary 8.5.3

$$\mathrm{mom}^k_t(\mathcal{E}\mathcal{S}^{\langle t\rangle}_c) = \frac{1}{N^k k!}{}_c\widetilde{e}_k(t) = \frac{-1}{N^{k-1}}\frac{c^{k+2}-1}{c^k}\mathrm{Eis}^k_{\mathbb{Q}_\ell}(t)$$

By Lemma 9.2.3 and 9.2.11 it suffices to compute the image of $\mathcal{E}\mathcal{S}^{\langle t\rangle}_c$ in $H^1(\infty, \underline{\mathrm{Hom}}_\infty(I(\mathbb{Z}_\ell(1))^{(n)}, \mathbb{Z}_\ell(k+2))$. By Lemma 9.2.10 and Lemma 9.2.8 this this image is equal to the image under $(*)$ of $\mathrm{mom}^k_{\zeta^b_N}(\mathcal{S}^{\langle\zeta^b_N\rangle}_c)$, which is by Proposition 4.2.2

$$\mathrm{mom}^k_\alpha(\mathcal{S}^{\langle\zeta^b_N\rangle}_c) = \frac{1}{k!N^k}\frac{c^{k+2}-1}{c^k}\widetilde{c}_{k+1}(\zeta^b_N).$$

This gives the desired formula

$$\mathrm{Dir}(\psi)) = \frac{-1}{Nk!}\sum_{b\in\mathbb{Z}/N\mathbb{Z}\setminus\{0\}}\psi(0,b)\widetilde{c}_{k+1}(\zeta^b_N) \in H^1(\infty, \mathbb{Q}_\ell(k+1)).$$

□

References

[Beĭ86] A. A. Beĭlinson. Higher regulators of modular curves. In *Applications of algebraic K-theory to algebraic geometry and number theory, Part I, II (Boulder, Colo., 1983)*, volume 55 of *Contemp. Math.*, pages 1–34. Amer. Math. Soc., Providence, RI, 1986.

[BK90] S. Bloch and K. Kato. L-functions and Tamagawa numbers of motives. In *The Grothendieck Festschrift, Vol. I*, volume 86 of *Progr. Math.*, pages 333–400. Birkhäuser Boston, Boston, MA, 1990.

[BL94] A. Beĭlinson and A. Levin. The elliptic polylogarithm. In *Motives (Seattle, WA, 1991)*, volume 55 of *Proc. Sympos. Pure Math.*, pages 123–190. Amer. Math. Soc., Providence, RI, 1994.

[Bla] D. Blasius. Motivic eisenstein. In *Pune conference*.

[Col98] P. Colmez. Théorie d'Iwasawa des représentations de de Rham d'un corps local. *Ann. of Math. (2)*, 148(2):485–571, 1998.

[Del77] P. Deligne. *Cohomologie étale*. Lecture Notes in Mathematics, Vol. 569. Springer-Verlag, Berlin, 1977. Séminaire de Géométrie Algébrique du Bois-Marie SGA 41$\o er$2, Avec la collaboration de J. F. Boutot, A. Grothendieck, L. Illusie et J. L. Verdier.

[Del89] P. Deligne. Le groupe fondamental de la droite projective moins trois points. In *Galois groups over $\mathbb{Q}$ (Berkeley, CA, 1987)*, volume 16 of *Math. Sci. Res. Inst. Publ.*, pages 79–297. Springer, New York, 1989.

[HK99] A. Huber and G. Kings. Degeneration of l-adic Eisenstein classes and of the elliptic polylog. *Invent. Math.*, 135(3):545–594, 1999.

[Hub] A. Huber. Cup-product construction. In *Pune conference*.

[Jan88] U. Jannsen. Continuous étale cohomology. *Math. Ann.*, 280(2):207–245, 1988.

[Kat93] K. Kato. Iwasawa theory and p-adic Hodge theory. *Kodai Math. J.*, 16(1):1–31, 1993.

[Kat04] K. Kato. p-adic Hodge theory and values of zeta functions of modular forms. *Astérisque*, (295):ix, 117–290, 2004. Cohomologies p-adiques et applications arithmétiques. III.

[Kin01] G. Kings. The Tamagawa number conjecture for CM elliptic curves. *Invent. Math.*, 143(3):571–627, 2001.

[Lan90] S. Lang. *Cyclotomic fields I and II*, volume 121 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1990. With an appendix by Karl Rubin.

[SGA72] *Groupes de monodromie en géométrie algébrique. I.* Lecture Notes in Mathematics, Vol. 288. Springer-Verlag, Berlin, 1972. Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969 (SGA 7 I), Dirigé par A. Grothendieck. Avec la collaboration de M. Raynaud et D. S. Rim.

[Sou81] C. Soulé. On higher p-adic regulators. In *Algebraic K-theory, Evanston 1980 (Proc. Conf., Northwestern Univ., Evanston, Ill., 1980)*, volume 854 of *Lecture Notes in Math.*, pages 372–401. Springer, Berlin, 1981.

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